$$L \quad ((R_1 + R_3)(R_2 + R_4) + R_5 (R_1 + R_2 + R_3 + R_4) \quad \ldots + R_3)(R_2 + R_4)]$$

MATRIX ALGEBRA

1.12. Matrix theory is a relatively recent mathematical development and acts as a powerful tool in various branches ot engineering and linear programming. Certain mathematical operations and results can be exrpessed in elegant and compact form by using matrix algebra. The study of matrices is motivated from the familiar problem of solving system of linear equations and linear transformations which often occur in the solution of engineering problems. Technique of abstract algebra (modern algebra) is a logical approach to study simultaneously the algebra of matrices and the geometry of linear transformations. As this approach is out of the scope in the persent book, a simple introduction to elementary operation and applicatiions of matrices is given in a direct way, just to form the basis for understanding an advanced course in matrix theory.

Notation and Definition of Matrix :

An ordered set of mn numbers (*elements*) arranged in a rectangular arrey of m rows and n columns and enclosed by a pair of brackets is called a matrix of *order* $m \times n$ (read as m by n), which is expressed in the form

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \cdots & \cdots \cdots \quad .. \\ \cdots & .. \cdots \quad .. \\ \cdots & \cdots .. \quad \cdots \\ a_{m1} & a_{m2} .. \quad a_{mn} \end{bmatrix} \qquad \ldots\ldots\ldots\ldots \quad (20)$$

[*Note* :– In indicating a matrix, parenthesis [ ] is used which makes it differ from a determinant where vertical parallel bars | | are used.]

The elements a_{ij} of a matrix may be either constant or functions. The first suffix of an element indicates the row and second indicates the column in which the element is located. Thus a_{ij} is an element of i^{th} row and j^{th} column.

Matrices as a whole are denoted by capital alphabets, while corresponding italic letters represent the elements of the matrix. Thus the above matrix in (20) can also be expressed in the form

$$\boxed{\begin{array}{l} A = [\, a_{ij} \,] \\ \text{where } i = 1, 2....m \\ \qquad\quad j = 1, 2....n \end{array}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

The other notation used to designate (i^{th}, j^{th}) elemnt is a^{i}_{j} where super–index shows the row and lower–index indicates the column. This notation is sometimes used in Tensor–analysis. Some times the following parenthesis

$$\{\ \},\quad (\ ),\quad \|\ \|$$

are used to denote matrices.

1·13. Types of Matrices :

(I) Column or row matrices

If a matrix is of order $m \times 1$ i. e. consists of single column with elements along m rows, the matrix is known as *column marix* or sometimes known as *column vector*. It can be expressed as

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

For convenience a column matrix is usually expressed by writing elements horizontally within curly brakets, thus

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = \{ a_{11}, a_{21}, \ldots am_1 \}$$

Similarly, a matrix of order $1 \times n$ i. e. with one row and elements along n columns, is known as *row matrix* or *row vector* which can be expressed as

$$A = [\, a_{11}, a_{12}, a_{13} \cdots a_{1n} \,]$$

(II) Zero or null matrix

A matrix, every element of which is zero, is known as zero or *null matrix* and it is denoted by Z. Thus

$$Z_{3\times4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(III) Transpose of a matrix

Matrix obtained by interchange of rows and columns is known as *tranpose* and is indicated by * or by a dash or by T. Thus if

$$A = [a_{ij}]$$

$$A^* = A^T = A' = [a_{ji}]$$ [star–notation is used in this book]

i. e. if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad A^* = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}$$

(IV) Square matrix and its determinant

A matrix of other $n \times n$ i. e. rows and columns are equal, is know as *square matrix* of order n. i. e.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot\cdot & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdot\cdot \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

A determinant of a square matrix, is such that its elements are same as the elements in the corresponding place of a square matrix A, and is denoted by | A | (read as determinant of A).

But determinant $|A|$ differs from its matrix A, in that it has a numerical value.

$$\boxed{\begin{array}{l}\text{If } |A| = 0, \text{ then A is called a } \textit{singular matrix} \\ \text{If } |A| \neq 0, \text{ then A is called a } \textit{non singular matrix}\end{array}} \quad \ldots\ldots (22)$$

(V) Symmetric and skew symmetric matrices :–

(i) The square matrix A is symmetric matrix, if its transpose is same as A i. e.

If $A = [a_{ij}]$, is symmetric then,

$$\boxed{A = A^*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

i. e. $[a_{ij}] = [a_{ji}]$ i. e. $a_{ji} = a_{ji}$.

Thus interchange of rows and columns does not change the form of matrix. Thus

$$\text{If } A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \qquad A^* = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

Since $A = A^*$, the matrix A is symmetric.

(ii) The matrix A is skew symmetric, if

$$a_{ji} = -a_{ji}$$

i. e. $(i . j)^{th}$ element of A is same as the element $(j, i)^{th}$ with sign changed.

Thus A is *skew symmetric* if

$$\boxed{A = -A^*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

From the definition, it follows that for diagonal elements

$$a_{ii} = -a_{ii}$$

$$\therefore\ 2a_{ii} = 0 \text{ or } a_{ii} = 0$$

i. e. *diagonal elements of skew symmetric matrix are zero.*

e. g. The matrix

$$A = \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix} \text{ is skew symmetric}$$

$$\text{as } a_{ij} = -a_{ji}$$

$$\text{or } A^* = \begin{bmatrix} 0 & 4 & -3 \\ -4 & 0 & -5 \\ 3 & 5 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$$

$$= -A$$

[This step is based on the product of a scalar with a matrix, explained in the operations of matrices].

(VI) Diagonal and scalar matrix

(i) If in a square matrix, all the elements except those along the diagonal, are zero, the matrix is known as *diagonal matrix*.

Thus the matrix $A = [a_{ij}]$ is a diagonal matrix, if

$$a_{ij} = 0 \text{ for } i \neq j$$

i. e. the matrix may be written as

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

(ii) If in a diagonal matrix, the elements along the diagonal are equal i. e. $a_{11} = a_{22} = .. = a_{nn} = k$, the matrix

$$A = \begin{bmatrix} k & 0 & 0 & .. & 0 \\ 0 & k & 0 & ... & 0 \\ 0 & 0 & k & ... & 0 \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & 0 & .. & k \end{bmatrix}$$

is known as a *scalar matrix.*

(**VII**) **Unit Matrix :**

A scalar matrix with $k = 1$, is called a *unit matrix* of order n and is denoted by I_n,i.e. Unit matrix is a diagonal matrix, all the diagonal elements as unity. Thus

$$I_n = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & .. & ... & .. \\ ... & .. & ... & ... \\ 0 & 0 & 0 & .. & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[*Note* :– Diagonal matrix, scalar matrix and unit matrix are all square matrices.]

The property of unit matrix I is that for a matrix A

$$\boxed{AI = A = IA} \quad \text{......... ....(25)}$$

. . . . in the multiplication of matrices.)

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(VIII) Adjoint of a Matrix :

If $A = [a_{ij}]$ be a *square* matrix, then the matrix obtained by replacing the element a_{ij} by the cofactor of a_{ji} of $|A|$, is known as adjoint of a matrix and is denoted as " adj. A". Thus

$$\text{adj. } A = [A_{ji}]$$

where A_{ij} = cofactor of a_{ij} in $|A|$ [By notation of determinant]

Thus to find the adj. A, first find the transpose of A i. e. A^* and replace the elements of A^* by the cofactors of the corresponding elements of the determinant of A^*.

Ex. Obtain the adjoint of matarix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 6 & -2 \\ 5 & 1 & 8 \end{bmatrix}$$

$$\text{Thus } A^* = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 6 & 1 \\ 3 & -2 & 8 \end{bmatrix} \text{ and } |A^*| = \begin{vmatrix} 2 & 4 & 5 \\ -1 & 6 & 1 \\ 3 & -2 & 8 \end{vmatrix}$$

Hence

adj. of A = matrix of cofactors of $|A^*|$

$$= \begin{bmatrix} 50 & 11 & -16 \\ -42 & 1 & 16 \\ -26 & -7 & 16 \end{bmatrix}$$

1.14. Operations of Matrices :

After defining various types of matrices, we now consider the fundamental operations viz. addition, subtraction and multiplication of matrix algebra.

(I) Equality of two matrices :

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal i. e.

$A = B$. if and only if

(i) A, B are of the same order $m \times n$

(ii) $a_{ij} = b_{ij}$ for every value of i and j

where $i = 1, 2, \ldots m$

$j = 1, 2, \ldots n$.

(iv) Identity Law of addition :

$$A + Z = A = Z + A$$

Where Z (null matrix) is the *identity* matrix under the operation of addition.

The proof of the above laws is left over to the students as an exercise.

IV) Multiplication of matrices :

The product AB of two matrices $A = [a_{ij}]$, and $B = [b_{ij}]$, is defined only on the assumption that *the columns of A are equal to the rows of B*. Thus the matrices A, B are conformable with respect to the product AB only if A is of order $m \times p$ and B is of order $p \times n$ and the product AB is $m \times n$ matrix $C = [c_{ij}]$, where

$$\boxed{\begin{aligned} c_{ij} &= a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \dots + a_{ip}\, b_{pj} \\ &= \sum_{k=1}^{p} a_{ik}\, b_{kj} \end{aligned}} \quad \dots (28)$$

for every $i = 1, 2, \dots m$ and every $j = 1, 2, \dots n$. Thus to con-
struct the element c_{ij} of the matrix product AB, multiply the
elements of i^{th} row of A with the corresponding elements of j^{th}
column of B and the sum of these products gives the $(i, j)^{th}$
element of AB. The technique for finding the $(i, j)^{th}$ element is
diagrammatically shown here, in order to write the product
of matrices directly.

$$A_{m\times p} \times B_{p\times n} = C_{m\times n}$$

$$\begin{array}{c} \\ \\ i^{th} \text{ row} \rightarrow \\ \\ \end{array} \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \\ a_{i1} & a_{i2} & \dots & a_{ip} \\ & & & \end{bmatrix} \begin{array}{c} \begin{array}{ccccc} & & \downarrow j^{th} \text{ column} & & \end{array} \\ \begin{bmatrix} \dots & \dots & b_{1j} & \dots & \dots \\ \dots & \dots & b_{2j} & \dots & \dots \\ \dots & \dots & \vdots & \dots & \dots \\ \dots & \dots & b_{pj} & \dots & \dots \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ \rightarrow \\ i \end{array} \begin{array}{c} \downarrow j \\ \begin{bmatrix} & \vdots & \\ \dots\dots & c_{ij} & \dots\dots \\ & \vdots & \end{bmatrix} \end{array}$$

where $c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \dots + a_{ip}\, b_{pj}$

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The above technique is useful in computing easily the product by using left index finger to run across the i^{th} row of A (left hand matrix) and simultaneously using the right index finger to run down the jth column of the matrix B (right hand matrix), multiplying elements in the corresponding positions and adding successively the products obtained. The following example will clarify the procedure.

Ex. Given $A = \begin{bmatrix} 5 & 6 & -1 & 4 \\ 3 & 5 & 2 & 1 \\ -1 & 0 & 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & -2 \\ 7 & 5 \\ 3 & -8 \end{bmatrix}$

find AB

$$AB = \begin{bmatrix} 5 & 6 & -1 & 4 \\ 3 & 5 & 2 & 1 \\ -1 & 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -2 \\ 7 & 5 \\ 3 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} (5)(2)+(6)(4)+(-1)(7)+(4)(3) & (5)(1)+(6)(-2)+(-1)(5)+(4)(-8) \\ (3)(2)+(5)(4)+(2)(7)+(1)(3) & (3)(1)+(5)(-2)+(2)(5)+(1)(-8) \\ (-1)(2)+(0)(4)+(3)(7)\times(6)(3) & (-1)(1)+(0)(-2)+(3)(5)+(6)(-8) \end{bmatrix}$$

$$= \begin{bmatrix} 39 & -44 \\ 43 & -5 \\ 37 & -34 \end{bmatrix}$$

By using definition of a product, we can evaluate for square matrix A

$A^n = A \times A \ldots n$ times (n is positive integer). where A is self conformable for the product.

Ex. 1. Given $A = \begin{bmatrix} & 6 \\ 4 & 7 \\ -3 & -8 \\ 5 & 11 \end{bmatrix}$, Evaluate A

$$AI = \begin{bmatrix} 2 & 6 \\ 4 & 7 \\ -3 & -8 \\ 5 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 7 \\ -3 & -8 \\ 5 & 11 \end{bmatrix}$$

$$= A$$

Ex. 2. Given $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \\ 5 & 6 & -3 \end{bmatrix}$, evaluate $A^3 - 3A^2 + A + 5I$

$$A^2 = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \\ 5 & 6 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \\ 5 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 18 & 11 \\ 16 & 25 & -5 \\ -11 & 9 & 38 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \\ 5 & 6 & -3 \end{bmatrix} \begin{bmatrix} 6 & 18 & 11 \\ 16 & 25 & -5 \\ -11 & 9 & 38 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 120 & 45 \\ -18 & 68 & 131 \\ 159 & 213 & -89 \end{bmatrix}$$

Hence

$$A^3 - 3A^2 + A + 51$$

$$= \begin{bmatrix} 49 & 120 & 45 \\ -18 & 68 & 131 \\ 159 & 213 & -89 \end{bmatrix} - 3 \begin{bmatrix} 6 & 18 & 11 \\ 16 & 25 & -5 \\ -11 & 9 & 38 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \\ 5 & 6 & -3 \end{bmatrix}$$

$$+ 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (49-18+2+5) & (120-54+3+0) & (45-33+1+0) \\ (-18-48-1+0) & (68-75+2+5) & (131+15+4+0) \\ (159+33+5+0) & (213-27+6+0) & (-89-114-3+5) \end{bmatrix}$$

$$= \begin{bmatrix} 38 & 69 & 13 \\ -67 & 0 & 150 \\ 197 & 192 & -201 \end{bmatrix}$$

In connection with the product of matrices, following points should be noted carefully :

(a) In general the matrix product AB is not *commutative* even if the product BA exists.

A is called *pre-factor* and B is called *pro-factor* in the product AB. In order that the product BA to exist, columns of B must be equal to rows of A. If A, B are square matrices, we can compute AB and $B\ A$, but the products will not represent the same matrix i. e. $AB \neq BA$.

Ex. 1. $A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ -1 & 0 & 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 5 & 7 \\ 6 & -1 \\ 2 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 23 & 30 \\ 31 & 6 \end{bmatrix}$$

$$BA = \begin{bmatrix} -3 & 1 \\ 5 & 7 \\ 6 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 0 & -7 \\ 3 & 15 & 26 & 25 \\ 13 & 18 & 3 & 19 \\ 2 & 6 & 8 & 18 \end{bmatrix}$$

Thus $AB \neq BA$

Ex. 2 $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -2 & 4 \\ 5 & 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 6 & 2 \\ 3 & 0 & 1 \\ 2 & -4 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -2 & 4 \\ 5 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 6 & 2 \\ 3 & 0 & 1 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 14 & 10 \\ 10 & -4 & 14 \\ 40 & 2 & 33 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 6 & 2 \\ 3 & 0 & 1 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & -2 & 4 \\ 5 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 34 & -4 & 42 \\ 14 & 5 & 10 \\ 13 & 16 & 7 \end{bmatrix}$$

Hence $AB \neq BA$

(b) If $AB = Z$, it is not necessarily true that either $A = Z$ or $B = Z$, where Z is a null matrix.

$$\text{For } AB = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -2 & 3 \\ 9 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & 10 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = Z$$

Here $A \neq Z$, $B \neq Z$.

(c) If $AB = AC$, it is not necessarily true that $B = C$ i. e. like ordinary algebra, the equal matrices in the identity cannot be cancelled. Thus

$$AB = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$$

Hence $AB = AC$ but $B \neq C$.

The matrix product satisfies following fundamental laws of algebra.

(i) *Associative law :*

Let $A_{m \times p} = [a_{ij}]$, $B_{p \times q} = [b_{ij}]$, $C_{q \times n} = [c_{ij}]$

then $A(BC) = (AB)C$

Proof:– Let $D = A(BC)$

$\therefore$ By definition of product

$$d_{ij} = \sum_{k=1}^{p} a_{ik} [BC]_{kj}$$

where $[BC]_{kj}$ is (k, j)th element of $[BC]$

$$= \sum_{k=1}^{p} a_{ik} \left(\sum_{r=1}^{q} b_{kr} c_{rj} \right)$$

$$= \sum_{k=1}^{p} \sum_{r=1}^{q} a_{ik}\, b_{kr}\, c_{rj} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(i)}$$

and let D′ = (AB) C, then

$$d'_{ij} = \sum_{r=1}^{q} [AB]_{ir}\, c_{rj}$$

$$= \sum_{r=1}^{q} \left(\sum_{k=1}^{p} a_{ik}\, b_{kr} \right) c_{rj}$$

$$= \sum_{k=1}^{p} \sum_{r=1}^{q} a_{ik}\, b_{kr}\, c_{rj} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(ii)}$$

Hence from (i) and (ii)

$$A\,(BC) = (AB)\,C.$$

(ii) *Distributive law :*

$$A\,(B+C) = AB + AC$$

or

$$(A+B)\,C = AC + BC$$

(*Note :* The relative position of matrices in simplification)

Proof :

$$[\, A_{m\times p},\ B_{p\times n} \text{ and } C_{p\times n} \,]$$

Let $D = [d_{ij}] = A\,(B + C)$

$$d_{ij} = \sum_{k=1}^{p} a_{ik}\, [\, B + C \,]_{kj}$$

$$= \sum_{k=1}^{p} a_{ik}\, (\, b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{p} a_{ik}\, b_{kj} + \sum_{k=1}^{p} a_{ik}\, c_{kj}$$

$= AB + AC$ [by def.]

(iii) $k\,(AB) = (kA)\,(B) = A\,(kB)$

Deduction from the laws :

(i) From associative law :

$A^2A = (AA)\,A = A\,(AA) = A^3$ **(denoted)**

Thus

A.A.A...m times $= A^m$ (m is a +ve integer)

$$\left.\begin{array}{l} A^m A^n = A^{m+n} \\ (A^m)^n = A^{mn} \end{array}\right\} \text{for } m, n \text{ as positive integers.}$$

(ii) From distributive law :

$$\begin{aligned}(A+B)\ (A-B) &= AA-AB+BA-BB\\ &= A^2-AB+BA-B^2\\ &\neq A^2-B^2 \quad [\text{ note this, as } AB \neq BA]\\ (A+B)^2 &= (A+B)\ (A+B)\\ &= A^2+AB+BA+B^2\\ &\neq A^2+2AB+B^2\end{aligned}$$

1·15. Inverse of a martix :

From the discussion of Laplace expansion of a determinant in art 1·4, we have

(i) The sum of the products of elements in one row (or column) with the corresponding cofactors of the elements of the same row (or column) is the value of the determinant.

(ii) The sum of the products of elements in one row (or column) with the corresponding cofactors of the elements from another row or column is zero.

Thus if the square matrix be $A = [a_{ij}]$ and A_{ij} denotes the cofactor of element a_{ij} in the determinant $|A|$, we have choosing elements from ith row of $|A|$

(i) $a_{i1} A_{i1} + a_{i2} A_{i2} + \ldots + a_{in} A_{in} = |A|$

(ii) $a_{i1} A_{j1} + a_{i2} A_{j2} + \ldots + a_{in} A_{jn} = 0 \ (i \neq j)$

The above results can be expressed as

$$\sum_{k=1}^{n} a_{ik} A_{jk} = \begin{array}{ll} |A| & \text{if } i = j \\ = 0 & \text{if } i \neq j \end{array} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

Or using a symbol δ_{ij} known as "kronecker's delta" defined as

$$\begin{aligned}\delta_{ij} &= 1 \quad (i = j)\\ &= 0 \quad (i \neq j).\end{aligned}$$

the result (29) can be compactly written as

$$\boxed{\sum_{k=1}^{n} a_{ik} A_{jk} = |A|\, \delta_{ij}} \quad \cdots \quad \cdots \quad \cdots \quad \cdots (30)$$

Now consider the product of square martix A and adj. A.
where $A = [a_{ij}]$ and adj. $A = [\alpha_{ij}]$
where α_{ij} = cofactor of element a_{ji} in $|A| = A_{ji}$

Let $D = A \ (\text{adj. } A) = [d_{ij}]$

$$\therefore \quad d_{ij} = \sum_{k=1}^{n} a_{ik} \cdot (\text{adj. } A)_{kj}$$

$$= \sum_{k=1}^{n} a_{ik}\, \alpha_{kj}$$

$$= \sum_{k=1}^{n} a_{ik} A_{jk} \quad [\alpha_{kj} = A_{jk} = \text{cofactor of } a_{jk} \text{ in } |A|]$$

$$= |A|\, \delta_{ij} \quad [\text{by result } 30]$$

Since $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$, the diagonal elements of matrix D are each $|A|$ and all others are zero. Hence

$$A.\ \text{adj. } A = D = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= |A|\, I$$

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Thus

$$\boxed{A.\ \text{adj. } A = |A|\ I} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

If a martix B for a given square martix A be such that

$$AB = I = BA$$

then B is known as inverse or reciprocal matrix of A and is denoted by A^{-1} (read as A–inverse). Thus A^{-1} satisfies the relation

$$\boxed{AA^{-1} = I = A^{-1}A} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

Now the result (31) can be written as

$$A\left(\frac{1}{|A|}\ \text{adj. } A\right) = I \quad (\text{as } |A| \text{ is a scalar})$$

comparing with (32), we have

$$\boxed{A^{-1} = \frac{1}{|A|}\ \text{adj. } A} \qquad \ldots \quad \ldots \quad \ldots \quad (33)$$

provided $|A| \neq 0$ i. e. *inverse of a square matrix A exists only if* $|A| \neq 0$ *i. e. the matrix A is non singular.*

Thus to find inverse of non-singular square matrix, find the adj. A by VIII in art. 1·13 and divide by | A .

Ex. Find A^{-1} for the matrix $A = \begin{bmatrix} & 3 & 1 \\ 4 & 5 & -3 \\ -1 & 6 & 7 \end{bmatrix}$

Step 1

$$\text{Transpose of } A = A^* = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 5 & 6 \\ 1 & -3 & 7 \end{bmatrix}$$

Step 2

adj. A = Matrix of cofactors of $|A^*|$

$$= \begin{bmatrix} 53 & -15 & -14 \\ -25 & 15 & 10 \\ 29 & -15 & -2 \end{bmatrix}$$

Step 3 $|A| = (2)\ (53) + (4)\ (-15) + (-1)\ (-14)$
$= 60$

$$\therefore\ A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{60}\begin{bmatrix} 53 & -15 & -14 \\ -25 & 15 & 10 \\ 29 & -15 & -2 \end{bmatrix}$$

Verification :–

$$AA^{-1} = \frac{1}{60}\begin{bmatrix} 2 & \text{[illegible]} & 1 \\ 4 & 5 & -8 \\ -1 & 6 & 7 \end{bmatrix}\begin{bmatrix} 53 & -15 & -14 \\ -25 & 15 & 10 \\ 29 & -15 & -2 \end{bmatrix}$$

$$= \frac{1}{60}\begin{bmatrix} 60 & 0 & 1 \\ 0 & 60 & 0 \\ 0 & 6 & 60 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

For non-singular matrix A, unique A^{-1} exists :–
If possible let A^{-1} and B be the inverses of A, then

$$A^{-1}A = AA^{-1} = I \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

and $$BA = AB = I \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i), premultiplying by B both the sides, we get

$$B\,(AA^{-1}) = BI$$

i. e. $(BA)\,A^{-1} = B$ (by associative law)

$IA^{-1} = B$ [from (ii)]

i. e. $A^{-1} = B$.

1·16. Important properties of matrices :

(a) *Transpose of a transpose is the matrix itself.*

If $A = [a_{ij}]$, then

$$(A^*)^* = A$$

This is clear from the definition of a transpose.

(b) *Transpose of a product is the product of transposes of matrices (in the product) taken in the reverse order* i. e.

$$(AB)^* = B^*A^*$$

Let $A_{m\times p} = [a_{ij}]$ and $B_{p\times n} = [b_{ij}]$

Then the product $C = AB$ is a matrix $m \times n$, where

$$c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$$

Now the matrices $A^*_{p\times m}$ and $B^*_{n\times p}$ are conformable if multiplied in order B^*A^* which is of order $n \times m$

Thus if c_{ji}^* is (j, i) th element of $(AB)^*$

then $\overset{*}{c}_{ji} = c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$

$$\sum_{k=1}^{p} b_{kj}\, a_{ik}$$

$$\sum_{k=1}^{p} (B^*)_{jk}\, (A^*)_{ki}$$

$$\boxed{\therefore \quad C^* = (AB)^* = B^*A^*} \qquad \dots \dots \dots \dots \quad (34)$$

Cor. 1. $(ABC\dots KL)^* = L^*K^*\dots C^*B^*A^*$

Cor. 2. $(A+A+\dots+L)^* = A^*+A^*+\dots+L^*$

Cor. 3. $(A^p)^* = (A^*)^p \dots (p \text{ a } +\text{ve integer})$

(c) *Every square matrix can be expressed as sum of symmetric and skew symmetric matrices.*

If A is a square matrix it can be expressed as

$$\boxed{A = \tfrac{1}{2}(A + A^*) + \tfrac{1}{2}(A - A^*)} \qquad \dots \dots \dots \quad (35)$$

where $A + A^*$ is symmetric and $A - A^*$ is skew symmetric for

$(A+A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$

$\therefore$ $A + A^*$ is symmetrie by result (23) and

$(A-A^*)^* = A^* - (A^*)^* = A^* - A = -(A - A^*)$

$\therefore$ $A - A^*$ is skew symmetric by result (24).

Ex. **Express the matrix** $\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix}$ **as a sum of symmetric and skew symmetric matrices.**

$$A^* = \begin{bmatrix} 2 & 14 & 3 \\ -4 & 7 & 5 \\ 9 & 13 & 11 \end{bmatrix}$$

$$\tfrac{1}{2}(A + A^*) = \tfrac{1}{2}\begin{bmatrix} 2+2 & 14-4 & 3+9 \\ -4+14 & 7+7 & 5+13 \\ 9+3 & 13+5 & 11+11 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix}$$

$$\text{and } \tfrac{1}{2}(A - A^*) = \tfrac{1}{2}\begin{bmatrix} 2-2 & -14-4 & -3+9 \\ 4+14 & 7-7 & 13-5 \\ 3-9 & 5-13 & 11-11 \end{bmatrix} = \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix}}_{\text{symmetric}} + \underbrace{\begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}}_{\text{skew-symmetric}}$$

(d) *Inverse of the product is the product of inverses of matrices (in the product) taken in the reverse order.*

i. e. $\boxed{(AB)^{-1} = B^{-1} A^{-1}}$ ... ... (36)

Pre-multiplying $B^{-1} A^{-1}$ by AB, we get

$$\begin{aligned} (AB)\ (B^{-1} A^{-1}) &= A\ (BB^{-1})\ A^{-1} \quad \text{(by associative law)} \\ &= A\ I\ A^{-1} \\ &= (AI)\ A^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

$$\therefore \quad (AB)\ (B^{-1} A^{-1}) = I$$

Hence by definition $(B^{-1}\ A^{-1})$ is inverse of AB

$$\therefore \quad (AB)^{-1} = B^{-1} A^{-1}$$

Generalising, we have

$$(ABC..KL)^{-1} = L^{-1} K^{-1} .. C^{-1}B^{-1}A^{-1}.$$

(e) *The operation of transposing and inversing are commutative :*

$\boxed{(A^{-1})^* = (A^*)^{-1}}$ .. .. (37)

We have

$$AA^{-1} = I$$

$$\therefore \quad (AA^{-1})^* = I^* = I$$

$$\therefore \quad (A^{-1})^* A^* = I$$

$\therefore$ By definition $(A^{-1})^*$ is inverse of A^*

i. e. $(A^*)^{-1} = (A^{-1})^*$

(*f*) *The determinant of the product of two matrices is the product of the determinants of the two matrices i. e.*

$$\boxed{|AB| = |A|\ |B|} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (38)$$

Let $A = [a_{ij}]$, $B = [b_{ij}]$ be sq. matrices of order n

$\therefore$ $|A| = |a_{ij}|$ and $|B| = |b_{ij}|$

Now $$AB = \left[\sum_{k=1}^{n} a_{ik}\, b_{kj}\right]$$

$$|A|\ |B| = \left|\sum_{k=1}^{n} a_{ik}\, b_{kj}\right|$$ [by rule of product of determinants in art. 1.6]

$$= |AB|$$

More generally

$$|ABC \ldots KL| = |A|\ |B|\ |C| \ldots |K|\ |L|.$$

(g) We have, from result (31)

$$A.\ \text{adj}.\ A = |A|\, I$$

$$= \begin{bmatrix} |A| & 0 & 0 & \ldots & 0 \\ 0 & |A| & 0 & \ldots & 0 \\ 0 & 0 & |A| & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & |A| \end{bmatrix}$$

which is nth order sq. matrix.

By the property (f) above

$$|A|\;|\text{adj. }A| = \begin{vmatrix} |A| & 0 & 0 & \ldots & 0 \\ 0 & |A| & 0 & \ldots & 0 \\ 0 & 0 & |A| & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & |A| \end{vmatrix}$$

$$= |A|^{n}$$

Hence

$$\boxed{|\text{adj. } A| = |A|^{n-1}} \qquad \ldots \quad \ldots \quad (39)$$

(h) **Orthogonal matrix** :—If a square matrix A satisfies the relation

$$AA^* = I$$

The matrix A is known as orthogonal matrix i. e.

$$\boxed{A^* = A^{-1}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

e.g. for matrix $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$AA^* = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus A is orthogonal martix

1·17. Elementary transformations :

Consider the set of equations.

$$\left.\begin{aligned} x - y + z &= 4 \\ 2x + 3y - z &= 1 \\ 3x - 2y + 4z &= 6 \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \quad \text{(i)}$$

By using definition of product, these can be expressed in matrix form

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \\ 3 & -2 & 4 \end{bmatrix} \quad \ldots \ldots \ldots \ldots \quad \text{(ii)}$$

is called the **coefficient matrix** of set of equation equations (i).

Let us consider the various formations of equation (i) by using following transformations :

(I) Interchange of positions :

Interchanging 1st. eqt. with the 3rd, we have equations (i) in the form

$$\left.\begin{aligned} 3x - 2y + 4z &= 6 \\ 2x + 3y - z &= 1 \\ x - y + z &= 4 \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad \text{(iii)}$$

with coef. matrix

$$\begin{bmatrix} 3 & -2 & 4 \\ 2 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \ldots \ldots \ldots \ldots \ldots \quad \text{(iv)}$$

(II) Multiplying one or more equations by a scalar :

Multiply 1st and 2nd equations of (i) by 2 and –3, the transformed equations are

$$\left.\begin{array}{rrrcr} 2x & -2y & +2z & = & 8 \\ -6x & -9y & +3z & = & -3 \\ 3x & -2y & +4z & = & 6 \end{array}\right\} \quad \dots \dots \dots \dots \quad (v)$$

with coef. matrix

$$\begin{bmatrix} 2 & -2 & 2 \\ -6 & -9 & 3 \\ 3 & -2 & 4 \end{bmatrix} \quad \dots \dots \dots \dots \dots \quad (vi)$$

(III) Multiplying one or more equations by a scalar and adding to the other :

Multiply 1st equation of (i) by 2 & 3 and add respectively to 2nd and 3rd eqtns. of (i), the transformed equations are

$$\left.\begin{array}{rrrcr} x & -\ y & +\ z & = & 4 \\ 4x & +\ y & +\ z & = & 9 \\ 6x & -5y & +\ 7z & = & 18 \end{array}\right\} \quad \dots \dots \dots \dots \quad (vii)$$

with coef. matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 1 \\ 6 & -5 & 7 \end{bmatrix} \quad \dots \dots \dots \dots \dots \quad (viii)$$

It can be shown, the equation (i) and equations (iii), (v) and (vii) obtained by transformations **I, II** and **III**, have the same solution i. e. equations (i), (iii), (v) and (vii) are known as equivalent. The corresponding coef. matrices representing corresponding equation, are called **equivalent matrices.**

Note that the equivalent matrices (ii), (iv), (vi) and (viii)

i. e.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & -2 & 4 \\ 2 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 & 2 \\ -6 & -9 & 3 \\ 3 & -2 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 1 \\ 6 & -5 & 7 \end{bmatrix}$$

are not *equal*

The column matrix of constants on the R. H. S. of equations, when associated with coef. matrix, the matrix thus formed is called *augmented matrix* and is denoted by

$$[A \mid C]$$

where C is column matrix of R. H. S. constants of the equations. Thus *augmented matrices* of equations (i). (iii) ,(v) and (vii) are

$$\left.\begin{matrix} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 2 & 3 & -1 & 1 \\ 3 & -2 & 4 & 6 \end{array}\right] & \left[\begin{array}{ccc|c} 3 & -2 & 4 & 6 \\ 2 & 3 & -1 & 1 \\ 1 & -1 & 1 & 4 \end{array}\right] \\ \left[\begin{array}{ccc|c} 2 & -2 & 2 & 8 \\ -6 & -9 & 3 & -3 \\ 3 & -2 & 4 & 6 \end{array}\right] & \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 4 & 1 & 1 & 9 \\ 6 & -5 & 7 & 18 \end{array}\right] \end{matrix}\right\} \quad \ldots \text{(ix)}$$

The operations I, II, III, which transform a matrix A into another matrix B, where algebraic equations represented by A, B have same solutions, are known as *elementary operations*.

Elementary transformation of matrix:

The tansformations I, II, III, discussed above can be directly applied to coef matrix or augmented matrix instead to the set of equations. This simplifies the computational labour to a great extent.

The *elementary row* (or column) operations are:

(i) *Interchange of ith row with jth row of a matrix and is denoted by* R_{ij} *and interchange of columns is represented by* C_{ij}.

(ii) *Multiplication of elements of ith row by non-zero k. It is denoted* R_i (k) *and for column it is represented by* C_i (k)

(iii) *Adding to the elements of ith row, the scalar multiples of corresponding elements in jth row and is represented by* R_{ij} (k) *and for column this is symbolised by* C_{ij} (k).

Consider the augmented maritx of equation (I)

$$A = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 2 & 3 & 1 & 1 \\ 3 & -2 & 4 & 6 \end{array}\right]$$

Interchanging 1st and 3rd row :

$$A \underset{R_{13}}{\sim} \left[\begin{array}{ccc|c} 3 & -2 & 4 & 6 \\ 2 & 3 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{array}\right]$$

Multiply 1st and 2nd rows of A by 2 and (−3)

$$A \underset{\substack{R_1(2) \\ R_2(-3)}}{\sim} \left[\begin{array}{ccc|c} 2 & -2 & 2 & 8 \\ -6 & -9 & 3 & -3 \\ 3 & -2 & 4 & 6 \end{array}\right]$$

Multiply 1st row of A by 2 and 3 and add respectively to 2nd and 3rd rows of A

$$A \underset{\substack{R_{21}(2) \\ R_{31}(3)}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 4 & 1 & 1 & 9 \\ 6 & -5 & 7 & 18 \end{array}\right]$$

These transformed matrices are same as given in (ix). The three elementary operations can also be performed on columns of a matrix.

These elementary operations reduce the computational labour of solving a system of linear equations, by transforming the augmented matrix by row-operations to the matrix where the first (r — 1) elements of rth row are zero.

Following steps are used to reduce the matrix by row transformation, to the form stated.

(i) Using operation II, reduce the element a_{11} to 1

(ii) Using operation III, reduce the elements $a_{21}, a_{31} \cdots a_{n1}$ to zero.

These operations are repeated for matrix, so the final form of matrix is

$$\begin{bmatrix} 1 & * & * & * & .. & * \\ 0 & 1 & * & * & .. & * \\ 0 & 0 & 1 & * & .. & * \\ .. & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

The method is illustrated in the followiug example.

Ex. Solve the system

$$\left.\begin{aligned} 6x_1+x_2+x_3&=-4\\ 2x_1-3x_2-x_3&=0\\ -x_1-7x_2-2x_3&=7 \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (i)$$

Here the augmented matrix is

$$A=\left[\begin{array}{ccc|c} 6 & 1 & 1 & -4\\ 2 & -3 & -1 & 0\\ -1 & -7 & -2 & 7 \end{array}\right]$$

Since 1 is required as element in 1st row and 1st column, we use R_{13}

$$A\sim\left[\begin{array}{ccc|c} -1 & -7 & -2 & 7\\ 2 & -3 & -1 & 0\\ 6 & 1 & 1 & -4 \end{array}\right]$$

Then using operation $R_1(-1)$

$$A\sim\left[\begin{array}{ccc|c} 1 & 7 & 2 & -7\\ 2 & -3 & -1 & 0\\ 6 & 1 & 1 & -4 \end{array}\right]$$

Using operations $R_{21}(-2)$ and $R_{31}(-6)$

$$A\sim\left[\begin{array}{ccc|c} 1 & 7 & 2 & -7\\ 0 & -17 & -5 & 14\\ 0 & -41 & -11 & 38 \end{array}\right]$$

To reduce element in 2nd row and 2nd column, we use the operation $R_2\left(-\frac{1}{17}\right)$

$$A\sim\left[\begin{array}{ccc|c} 1 & 7 & 2 & -7\\ 0 & 1 & \frac{5}{17} & -\frac{14}{17}\\ 0 & -41 & -11 & 38 \end{array}\right]$$

Using $R_{32}(41)$, we get

$$A\sim\left[\begin{array}{ccc|c} 1 & 7 & 2 & -7\\ 0 & 1 & \frac{5}{17} & -\frac{14}{17}\\ 0 & 0 & \frac{18}{17} & \frac{72}{17} \end{array}\right]$$

Thus the equivalent set of equations are

$$x_1+7x_2+2x_3=-7$$

$$x_2+\frac{5}{17}x_3=-\frac{14}{17}$$

$$\frac{18}{17}x_3=\frac{72}{17}$$

$\therefore$ Thus the solutions are

$$x_3=4$$
$$x_2=-2$$
$$x_1=-1$$

1·18. The rank of matrix :-

A *minor* of order r of a matrix $A_{m \times n}$ [where $r \leq \min(m,n)$] is a determinant of rth order formed by elements situated at the intersections of r rows and columns of the matrix A, in their natural order.

The matrix is said to be of *rank* r if

(i) At least one minor of order r is *non-vanishing* and

(ii) Every minor of order $(r+1)$ *vanishes*.

Since for a minor of higher rank than $(r+1)$ will involve vanishing minor of order $(r+1)$ [by condition (ii)], we have the condition that every minor of A of order greater than or equal to $(r+1)$ is zero. Thus the alternative definition is that the rank of matrix A is the *maximum* order of its *non-vanishing minor* and it is denoted as

$$\rho(A) = r$$

Note :- (i) If there exists a non-vanishing minor of order k of A, then $\rho(A) \geq k$

(ii) If all minors of A of order $(k+1)$ are zero, then

$$\rho(A) \leq k$$

Ex. Find the ranks of the matrices :

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

For (i)

$$|A| = 0, \text{ But } \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$$

$\therefore$ Non vanishing minor is of order 2

$\therefore$ $\rho(A) = 2$

(ii) For this matrix, all minors of order 3 are zero

$$\text{but } \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = -3 \neq 0$$

Hence $\rho(A) = 2$.

Elementary transformations which reduce matrix A to matrix B does not affect the *rank* of matrix

i. e. A and B have the same rank.

Thus matrices A, B which have same rank are called *equivalent matrices*.

Proof : Let r and r' be the ranks of A and B which is obtained by one of the row operations and consider a square sub-matrix B_{r+1} of order $r+1$, whose rows uniquely correspond to those of submatrix A_{r+1} of A depending upon the nature of elementary row operation.

(I) For the operation R_{ij} :—

The rows of B_{r+1} will differ from A_{r+1} in relative position, hence

$$\left.\begin{aligned} |B_{r+1}| &= |A_{r+1}| \\ \text{or } &= -|A_{r+1}| \end{aligned}\right\} \quad \ldots \quad .. \quad \ldots \quad \text{(i)}$$

(II) For $R_j\ (k)$:

The rows of B_{r+1} uniquely detemine A_{r+1} which may or may not contain row affected by the transfomation, hence

$$\left.\begin{aligned} |B_{r+1}| &= |A_{r+1}| \\ \text{or } &= k\ |A_{r+1}| \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \text{(ii)}$$

(III) For $R_{ij}\ (k)$: –

The row of A_{r+1} corresponding to B_{r+1} may or may not contain ith row which is affected by the transformation.

For the affected ith row

element of B_{r+1} in (i, p) th position

Hence $= a_{ip} + ka_{jp}$

$$\left.\begin{aligned} |B_{r+1}| &= |A_{r+1}| \\ \text{or } &= |A_{r+1}| + k\ |C_{r+1}| \end{aligned}\right\} \quad \ldots \quad .. \quad \text{(iii)}$$

where $|C_{r+1}|$ is minor of $(r+1)$ th order of A

As A is of rank r, we have

$|A_{r+1}| = 0$ and $|C_{r+1}| = 0$.

Thus for all these elementary transformations, we have from (i), (ii), (iii)

$$|B_{r+1}| = 0.$$

Hence

$$r' \leqslant r \text{ [by [illegible] given]} \quad \ldots \quad \ldots \quad \text{(iv)}$$

Similarly, it is possible to obtain A by elementary operations on B and using similar reasoning as above, we get

$$r \leq r' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(v)}$$

Hence from (iv) and (v)

$$r' = r.$$

Thus the elementary transformation of rows (or columns) does not affect the rank of matrix.

1. 19. Echelon or Normal matrix

The *cannonical* or *echelon* form of matrix A is a row equivalent matrix C of rank r with following properties.

(i) One or more elements in each of the first r rows are non zero and the elements in the remaining rows are zero.

(ii) In the first r rows, the first non-zero element in each row is 1 and it appears in a column to the right of the first non-zero element of the preceding row.

For example, a matrix of order 5×8 and of rank 3, the echelon form is

$$\begin{bmatrix} 0 & 0 & 1 & 4 & -9 & 2 & 3 & 6 \\ 0 & 0 & 0 & 1 & 5 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the rank of matrix being three

(i) first three rows contain at least one non zero element and other (i. e. 4th and 5th) rows are zero

(ii) in the first three rows, first non-zero element is 1 and the column in which it is present is to the right to the column in which 1 is present in the preceding row.

Any matrix A can be reduced by row operations to echelon form by the following steps.

(i) Use the transformation R_{ij}, if, the element a_{11} of A is zero. The interchange of rows of A to be effected such that elements in the columns on the left of 1st non zero element of 1st row, must be zero.

(ii) If the first non-zero element in the first row is a_{ij} use $R_i\left(\frac{1}{a_{ij}}\right)$ to reduce the element to 1

(iii) Using $R_{ji}(k)$ with appropriate multiple k, reduce all elements in 'j' column [i. e. the column of a_{ij}] to zero.

Applying the above procedure to the successive rows, we can reduce A to a cannonical form. This is illustrated by the following example :-

Ex. Obtain cannonical matrix row equivalent to the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 3 & 5 & 1 & 2 \\ 5 & -1 & 2 & 2 \\ 2 & 6 & 5 & 3 \\ 1 & 3 & -3 & -1 \end{bmatrix}$$

Since 0 is the element in 1 st row, and 1 is present in 5th row, we have

$$A \underset{R_{15}}{\sim} \begin{bmatrix} 1 & 3 & -3 & -1 \\ 3 & 5 & 1 & 2 \\ 5 & -1 & 2 & 2 \\ 2 & 6 & 5 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} \underset{\substack{R_{21}(-3) \\ R_{31}(-5) \\ R_{41}(-2)}}{\sim} \begin{bmatrix} 1 & 3 & -3 & -1 \\ 0 & -4 & 10 & 5 \\ 0 & -16 & 17 & 7 \\ 0 & 0 & 11 & 5 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

$$\underset{R_{25}}{\sim} \begin{bmatrix} 1 & 3 & -3 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & -16 & 17 & 7 \\ 0 & 0 & 11 & 5 \\ 0 & -4 & 10 & 5 \end{bmatrix} \underset{\substack{R_{32}(8) \\ R_{52}(2)}}{\sim} \begin{bmatrix} 1 & 3 & -3 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 25 & 15 \\ 0 & 0 & 11 & 5 \\ 0 & 0 & 12 & 7 \end{bmatrix}$$

$$\underset{\substack{R_{35}(-2) \\ R_{45}(-1)}}{\sim} \begin{bmatrix} 1 & 3 & -3 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 12 & 7 \end{bmatrix} \underset{\substack{R_{43}(1) \\ R_{53}(-12)}}{\sim} \begin{bmatrix} 1 & 3 & -3 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\underset{R_{54}(-5)}{\sim}\begin{bmatrix} 1 & 3 & -3 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is the echelon form.

To find the rank of a matrix from the definition, several determinants are required to be evaluated. Thus computational work is very much reduced by elementary row (or column) operations which reduces the matrix to the echelon form which has the same rank as the given matrix. This form enables us to find the maximum non-vanishing minor i. e. gives the rank of matrix.

Thus to determine the rank of matrix $A_{m \times n}$ reduce it to echelon form and if k rows contain the all zero elements then the rank of matrix $= m - k$

By using row and column transformations on the matrix A it can be reduced to one of the forms

$$I_r, \quad \begin{bmatrix} I_r & Z \\ Z & Z \end{bmatrix}, \quad [\,I_r, Z\,], \quad \begin{bmatrix} I_r \\ Z \end{bmatrix}$$

which are known as *normal forms* of a matrix.

For example (r is a rank)

(i) For $A_{3 \times 3}$ and $r = 3$

$$\text{Normal form of } A_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

(ii) For $A_{4 \times 5}$ and $r = 2$

$$\text{Normal form} = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} I_2 & Z \\ \hline Z & Z \end{array}\right]$$

(iii) For $A_{3\times5}$ and $r = 3$

$$\text{Normal form} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] = [I_3 \mid Z]$$

(iv) For $A_{5\times3}$ and $r = 3$

$$\text{Normal form} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\frac{I_3}{Z}\right]$$

Ex. Find non-singular matrices P, Q so that PAQ is a normal form where

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Now

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ (R_{13})$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} R_2 - R_1(-3) \\ R_3 - R_1(-4) \end{array} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ \text{[illegible]} & & \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\left\{ \begin{array}{l} C_2 - C_1(-1) \\ C_3 - C_1(-1) \\ C_4 - C_1(-2) \end{array} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 6 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\left\{ \begin{matrix} R_{23} \\ R_2(-1) \\ R_3(-1) \end{matrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & -28 & -56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 6 & -1 & -9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\{ R_3 - R_2(-6) \}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ \frac{-3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left\{ \begin{matrix} C_3 - C_2(-5) \\ C_4 - C_2(-10) \end{matrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ \frac{-3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$N = PAQ$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ \frac{-3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1·20 Inverse of a matrix by elementary trnasformations

Since the normal of non-singular square matrix A is a unit matrix A, we reduce A by non singular matrices P, Q to a normal form i. e.

$$PAQ = I$$

Pre multiplying by P^{-1} and post multiplying by Q^{-1}, we have

$$A = P^{-1} I Q^{-1} = P^{-1} Q^{-1} \quad (PP^{-1} = QQ^{-1} = I)$$

$$A^{-1} = (P^{-1} Q^{-1})^{-1} = QP \quad \ldots. \quad (i)$$

Similary A can be reduced to normal form (unit matrix) by only row transformations.

Thus $$PA = I$$

Post multiplying by A^{-1}, we get

$$P = I\,A^{-1} = A^{-1} \qquad \ldots\ldots \quad \text{(ii)}$$

Hence by using elementary transformations we get A^{-1} by (i) or (ii)

Ex. Using row transformations find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

We have

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A. \ R_{13}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix} A, \ \begin{matrix} R_2 - R_1(-4) \\ R_3 - R_1(-2) \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/5 & +4/5 \\ -1 & 0 & 2 \end{bmatrix} A, \ \begin{matrix} R_2\,(-1/5) \\ R_3\,(-1) \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/5 & 4/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} A, \ R_3 - R_2\,(-1)$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4/5 & -19/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} A, \ \begin{matrix} R_1 - R_3\,(-4) \\ R_2 - R_3\,(-3) \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} A$$

$$\therefore\ A^{-1} = P = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

1·21. Linear Equations :

One of the most frequent applications of matrices to various fields in engineering and social sciences arises from the need to solve a system of linear equations.

Consider a system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=d_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=d_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=d_m \end{array}\right\} \quad \ldots \ldots \quad \text{(i)}$$

Here the mn scalars a_{ij} and m scalars d_i are fixed. The solution of the set of equations is any set of values of $x_1, x_2, \ldots x_n$ which satisfy simultaneously the m-equations in (i). To solve the system is to find all possible solutions.

The system (i) can be written in compact form by using matrix notation :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

or more compactly, as

$$AX = D \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

If $D = Z$ (null matrix), the system of equations

$$AX = Z \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

is known as *system of homogeneous equations.*

When the system (ii) or (iii) has a solution, it is said to be *consistent,* otherwise the system is called *inconsistent*. A consistent set of equations may have one solution or infinitely many solutions.

The system of equations is consistent if the coef. matrix A and the augmented matrix [A | D] have the same rank.

Homogeneous Equations :-

$$AX = Z$$

This system is *always consistent* as the rank of matrix A and the augmented matrix [A, Z] are the same.

Thus $X = Z$ i. e. $x_1 = x_2 = \ldots = x_n = 0$ is always a solution and is known as *trivial solution*.

The other forms of solutions depend on the rank r of matrix A :

(i) If $r = n$, then there is a unique solution which is a *trivial solution*.

(ii) The other solutions besides trivial solution exist if $r < n$ and in this case there exists r solutions, where r unknowns are expressed as linear combinations of $n-r$ remaining unknowns.

Ex. Find the non-trivial solutions of

$$\text{(i)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 0 \\ 4x_1 + 5x_2 + 4x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

$$\text{(ii)}\quad \begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 6x_3 &= 0 \end{aligned}$$

(i) The augmented matrix is

$$[A, Z] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 0 \\ 4 & 5 & 4 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \underset{\substack{R_{21}(-2) \\ R_{31}(-4) \\ R_{41}(-1)}}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -3 & -8 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$$

$$\underset{R_2(-1)}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -3 & -8 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right] \underset{\substack{R_{32}(3) \\ R_{42}(1)}}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_3\left(\frac{1}{7}\right)}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ The rank of matrix [A, Z] $= 4 - 1 = 3$

$= n =$ number of variables.

$\therefore$ The system has unique solution as trivial solution and no other.

(ii) Consider the augmented matrix

$$[A, Z] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & 5 & 6 & 0 \end{array}\right] \underset{R_{21}(-2)}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ [illegible] & & & \end{array}\right] \quad \ldots \ldots (a)$$

Thus the rank of matrix is $2 < 3$, the number of variables and thus possess a solution other than trivial solution from reduced matrix (*x*), the equivalent system of equations is

$$x_1 - 2x_2 + x_3 = 0$$
$$9x_2 = 0$$

Thus infinite set of solutions are given by

$$x_1 = -c$$
$$x_2 = 0$$
$$x_3 = c$$

for set of values for parameter c.

Note :- A system of n homogeneous equations in n variables possesses a solution other than trivial solution, if coef. matrix $A_{n \times n}$ is singular i. e

$$|A| = 0$$

i. e. the rank r of matrix A is less than n.

Non-Homogeneous Equations :

Consider a system of m equations in n-unknowns given by

$$AX = D$$

(I) $m \neq n$: The system possesses a solution or is consistent if the coefficient matrix A and augmented matrix [A | D] have the same rank.

If r is the rank of these matrices, then r unknowns can be expressed in terms of remaining $n - r$ unknowns to which whatever values can be assigned.

(II) $m = n$: A system of n non-homogeneous equations in n unknowns has a unique solution, provided the rank of matrix A is n i. e. [A] is a non-sigular matrix i. e.

$$|A| \neq 0$$

One method by using determinants is discussed in art. 1·8

The other method by using matrix form is as follows :

The equations are

$$AX = D \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

Since A is a matrix of order $n \times n$ and $|A| \neq 0$, A^{-1} xists. Premultiply both sides of (i) by A^{-1}, we have

$$A^{-1}(AX) = A^{-1}D$$

i. e. $(A^{-1}A)X = A^{-1}D$ (by associative law)

i. e. $IX = A^{-1}D$ [as $AA^{-1} = I$]

$$\therefore \quad X = A^{-1}D$$

which gives the required solution.

Ex. Solve the equations

$$\text{(i)}\quad \begin{aligned} x_1 - x_2 + x_3 - x_4 + x_5 &= 1\\ 2x_1 - x_2 + 3x_3 + 4x_5 &= 2\\ 3x_1 - 2x_2 + 2x_3 + x_4 + x_5 &= 1\\ x_1 + x_3 + 2x_4 + x_5 &= 0 \end{aligned}$$

$$\text{(ii)}\quad \begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5\\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2\\ 4x_1 + 5x_2 + 3x_3 &= 7 \end{aligned}$$

$$\text{(iii)}\quad \begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 11\\ x_1 + 5x_2 + 7x_3 &= 15\\ 3x_1 + 11x_2 + 13x_3 &= 25 \end{aligned}$$

(i) Augmented matrix is

$$[A, D] = \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1\\ 2 & -1 & 3 & 0 & 4 & 2\\ 3 & -2 & 2 & 1 & 1 & 1\\ 1 & 0 & 1 & 2 & 1 & 0 \end{array}\right].$$

$$\underset{\substack{R_{21}(-2)\\ R_{31}(-3)\\ R_{41}(-1)}}{\sim} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1\\ 0 & 1 & 1 & 2 & 2 & 0\\ 0 & 1 & -1 & 4 & -2 & -2\\ 0 & 1 & 0 & 3 & 0 & -1 \end{array}\right]$$

$$\underset{\substack{R_{32}(-1)\\ R_{42}(-1)}}{\sim} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1\\ 0 & 1 & 1 & 2 & 2 & 0\\ 0 & 0 & -2 & 2 & -4 & -2\\ 0 & 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$$\underset{R_3(-\frac{1}{2})}{\sim} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1\\ 0 & 1 & 1 & 2 & 2 & 0\\ 0 & 0 & 1 & -1 & 2 & 1\\ 0 & 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$$\underset{R_{43}(1)}{\sim} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1\\ 0 & 1 & 1 & 2 & 2 & 0\\ 0 & 0 & 1 & -1 & 2 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\alpha)$$

Thus t e rank of coef. matrix A = 3

= rank of augmented matrix [A | D]

Thus the system is consistent. The system of equivalent equations obtained from cannonical matrix (α) is

$$\left.\begin{aligned} x_1 - x_2 + x_3 - x_4 + x_5 &= 1\\ x_2 + x_3 + 2x_4 + 2x_5 &= 0\\ x_3 - x_4 + 2x_5 &= 1 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\beta)$$

Treating x_4, x_5 as arbitrary, we express x_1, x_2, x_3 in terms of x_4, x_5. Thus the infinity solutions of the system are obtained by substituting x_3 ($= 1+x_4 - 2x_5$) from 3rd equation into 2nd equation of (β) which gives x_2 in terms of x_4 and x_5. Substitute the values of x_2, x_3 thus obtained in 1st equation of (β) to get x_1 in terms of x_4, x_5. Thus for solutions, we have

$$x_1 = -1 - 3x_4 + x_5$$

$$x_2 = -1 - 3x_4$$

$$x_3 = 1 + x_4 - 2x_5$$

where x_4, x_5 are arbitrary.

(ii) The augmented matrix is

$$[A \mid D] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{array}\right]$$

$$\underset{\substack{R_{21}(-2) \\ R_{31}(-4)}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -5 & -4 & -13 \end{array}\right]$$

$$\underset{R_{32}(-1)}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

Here the rank of coef. matrix $\sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is 2

and the rank of augmented matrix $\sim \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$ is 3

Hence as rank of coef. matrix ($r = 2$) is not equal to the rank ($r = 3$) of the augmented matrix, the system of equations is inconsistent and hence no solution exists.

(iii) This is the case of $m = n$. Hence for coef. matrix A

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{vmatrix} = -16 \neq 0$$

Hence the system is consistent.

The matrix form is

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 25 \end{bmatrix}$$
$$\quad A \qquad\qquad X \qquad\qquad D$$

For A^{-1} : $A^* = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 5 & 11 \\ 4 & 7 & 13 \end{bmatrix}$

$\therefore$ adj. A = matrix of cofactors of $|A^*|$

$$= \begin{bmatrix} -12 & 5 & 1 \\ 8 & 14 & -10 \\ -4 & -13 & 7 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj. } A = -\frac{1}{16} \begin{bmatrix} -12 & 5 & 1 \\ 8 & 14 & -10 \\ -4 & -13 & 7 \end{bmatrix}$$

$$\therefore \quad X = A^{-1} D = -\frac{1}{16} \begin{bmatrix} -12 & 5 & 1 \\ 8 & 14 & -10 \\ -4 & -13 & 7 \end{bmatrix} \begin{bmatrix} 11 \\ 15 \\ 25 \end{bmatrix}$$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{16} \begin{bmatrix} -32 \\ 48 \\ -64 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$

$\therefore$ Solution is

$x_1 = 2$, $x_2 = -3$, and $x_3 = 4$.

·22 Linear dependence and linear independence of vectors

A set of vectors $\mathbf{X}_1\ \mathbf{X}_2 \ldots \mathbf{X}_n$, is said to be linearly independent set if every relation of type

$$C_1 \mathbf{X}_1 + C_2 \mathbf{X}_2 + \ldots\ldots + C_n \mathbf{X}_n = \mathbf{O} \quad \ldots\ldots\ldots \quad (i)$$

where **O** is zero column matrix, implies

$$C_1 = C_2 \quad \ldots\ldots\ldots = C_n = 0$$

where $C_1, C_2 \ldots C_n$ are scalars.

If there exist scalers $C_1\ C_2 \ldots C_n$ for relation (i) such that not *all* are zero, then vectors $\mathbf{X}_1\ \mathbf{X}_2 \ldots \mathbf{X}_n$ are linearly dependent set.

Ex. State wether following set of vectors are linearly dependent or are independent. If dependent find the relation between them.

(i) $X_1 = (1, 2, 3)^T, X_2 = (3, -2, 1)^T$

$X_3 = (1, -6, -5)^T$

(ii) $X_1 = (1, 2, 4)^T, X_2 = (3, 7, 10)^T$

(i) $C_1 X_1 + C_2 X_2 + C_3 X_3 = O$ gives

$$C_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i. e. the equations to determine C_1, C_2, C_3 are (hom eqt.).

$$C_1 + 3C_2 + C_3 = 0$$
$$2C_1 - 2C_2 - 6C_3 = 0$$
$$3C_1 + C_2 - 5C_3 = 0$$

which are homegeneous having a trivial solutions $C_1 = C_2 = C_3 = 0$ For non-trivial solution consider the rank of coeficient matrix is.

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & -8 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix $= 2$ (< 3 no of variables). Hence non trivial solution exists. Thus from last matrix, we get

$$C_2 + C_3 = 0, \; C_1 + 3C_2 + C_3 = 0$$

i. e. $C_3 = -\lambda \; C_2 = \lambda, C_1 = -2\lambda$

Thus set of vectors are linearly dependent and the relation between them is

$$-2X_1 + X_2 - X_3 = O$$

(ii) $C_1 X_1 + C_2 X_2 = 0$ gives

$$C_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the set of equations for C_1 and C_2 is

$$C_1 + 3C_2 = 0, \; 2C_1 + 7C_2 = 0, \; 4C_1 + 10C_2 = 0$$

which has a trivial solution $C_1 = C_2 = 0$ To find whether non trivial solution exists, the coef, matrix is

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 4 & 10 \end{bmatrix} \sim \begin{bmatrix} : & 3 \\ 0 & 1 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Rank of the matrix = 2 = no of variables No non-trivial solution exist and thus there is only unique solution $C_1 = C_2 = 0$ Vectors $\mathbf{X}_1, \mathbf{X}_2$ are linearly independent.

1.23 Linear transformations :-

• Let the homogeneous equations of linear transformation from (x_1, x_2) variables to (y_1, y_2) variables be given by

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 \\ y_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

i. e. in matrix notation it is given by

$$Y = A\mathbf{X} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43a)$$

$$\text{where } Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Similarly let the homogeneous linear transform equations from variables (y_1, y_2) to variables (z_1, z_2) be

$$\left.\begin{aligned} z_1 &= b_{11}y_1 + b_{12}y_2 \\ z_2 &= b_{21}y_1 + b_{22}y_2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

i. e. in matrix form, we have

$$Z = BY \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44a)$$

$$\text{where} \quad Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then the resultant transformation from $\mathbf{X}$ to Z, is given by

$$\begin{aligned} Z &= BY \\ &= B(A\mathbf{X}) \\ &= (BA)\mathbf{X} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (45) \end{aligned}$$

(by associative law)

The equation (45) represents the resultant of transformations (43) and (44), where A, B are known as transformation matrices of transformations (43) and (44).

Note :— The linear transformation from $\mathbf{X}$ to Y i. e.

$$Y = A\mathbf{X}$$

determines uniquely Y for given **X**. However the converse is true, if A is non-singular for

$$\mathbf{X} = A^{-1} Y$$

so that to given value of Y corresponds a unique **X**. In this case the transformation is said to have one to one correspondence.

Solved Problems :

Example 1. Given $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & -1 \end{bmatrix}$

Verify :- (I) $(A+B)^2 = A^2+AB+BA+B^2$

(II) $(AB)^* = B^*A^*$

(III) $(AB)^{-1} = B^{-1} A^{-1}$

(IV) $(A^2)^* = (A^*)^2$

We have

$$A^* = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & -2 \\ -1 & 4 & 3 \end{bmatrix} \quad B^* = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \quad \dots \dots \dots \quad \text{(i)}$$

$$A^2 = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 6 & 3 & -2 \\ 5 & 3 & -2 \end{bmatrix} \quad \text{..(ii)}$$

$$B^2 = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ -7 & 6 & -4 \\ -3 & 3 & -2 \end{bmatrix} \quad \text{..(iii)}$$

$$AB = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -4 & 1 \\ 3 & -4 & 2 \end{bmatrix} \quad \text{.. (iv)}$$

$$BA = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -7 & 8 \\ -1 & -3 & 3 \end{bmatrix} \quad \text{.. (v)}$$

adj A = Matrix of cofactors of $| A^* |$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 6 & 3 & -2 \\ 5 & 3 & -2 \end{bmatrix} \quad \dots \dots \dots \dots \dots \quad \text{(vi)}$$

adj. B = Matrix of cofactors of $|B^*|$

$$= \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad \text{(vii)}$$

$|A| = 1$ and $|B| = 1$ .. .. **(viii)**

(I) $A + B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 3 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(A+B)^2 = (A+B)\,(A+B) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 3 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 3 \\ 4 & -1 & 1 \end{bmatrix} \quad \ldots \ldots \ldots \quad (\alpha)$$

and from (ii), (iii), (iv) and (v), we have

$$A^2 + AB + BA + B^2 = \begin{bmatrix} -1 & -1 & 1 \\ 6 & 3 & -2 \\ 5 & 3 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ 3 & -4 & 1 \\ 3 & -4 & 2 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 2 & -2 \\ 1 & -7 & 8 \\ -1 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 1 \\ -7 & 6 & -4 \\ -3 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 3 \\ 4 & -1 & 1 \end{bmatrix} \quad \ldots \ldots \ldots \ldots \ldots \quad (\alpha')$$

Thus from (α) and (α'), we get

$$(A+B)^2 = A^2 + AB + BA + B^2$$

(Note $AB \neq BA$)

(II) $(AB)^* = \begin{bmatrix} -1 & [illegible] & [illegible] \\ [illegible] & [illegible] & [illegible] \\ [illegible] & [illegible] & [illegible] \end{bmatrix}$ .. .. (β)

$$B^*A^* = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & -2 \\ -1 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 3 \\ 1 & -4 & -4 \\ 0 & 2 & 2 \end{bmatrix} \quad .. \quad .. \quad .. \quad .. \quad (\delta')$$

Thus from (δ) and (δ') we get

$$(AB)^* = B^*A^*$$

(III) $(AB)^* = \begin{bmatrix} -1 & 3 & 3 \\ 1 & -4 & -4 \\ 0 & 1 & 2 \end{bmatrix}$

adj $(AB)^* = \begin{bmatrix} -4 & -2 & 1 \\ -3 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ and $|AB| = 1$

Thus

$$(AB)^{-1} = \frac{1}{|AB|} \text{ adj. } (AB)$$

$$= \begin{bmatrix} -4 & -2 & 1 \\ -3 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad .. \quad .. \quad .. \quad .. \quad (\epsilon$$

From (vi), (vii) and (viii)

$$A^{-1} = \frac{1}{|A|} \text{ adj.} A = \begin{bmatrix} -1 & -1 & 1 \\ 6 & 3 & -2 \\ 5 & 3 & -2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{ adj. } B = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore \quad B^{-1}.A^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 6 & 3 & -2 \\ 5 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -2 & 2 \\ -3 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix} = (AB)^{-1} \text{ [from } (\epsilon)]$$

(IV) From (i) and (ii)

$$(A^2)^* = \begin{bmatrix} -1 & 6 & 5 \\ -1 & 3 & 3 \\ 1 & -2 & -2 \end{bmatrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (\rho)$$

$$\text{and } (A^*)^2 = A^* \cdot A^* = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & -2 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & -2 \\ -1 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 6 & 5 \\ -1 & 3 & 3 \\ 1 & -2 & -2 \end{bmatrix}$$

$$= (A^2)^* \; [\text{ from result } (\rho)\,]$$

Example 2. Show that

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}, \text{ where } n \text{ is a positive integer}$$

The proof is by induction

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix} \text{ for } n = 2$$

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^2$$

$$= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix} \text{ for } n = 3$$

Thus result being true for $n = 2$, $n = 3$, let us assume it to be true for $n = m$.

Hence

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^m = \begin{bmatrix} \lambda^m & m\lambda^{m-1} \\ 0 & \lambda^m \end{bmatrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(i)}$$

Now

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^{m+1} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^m$$

$$= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^m & m\lambda^{m-1} \\ 0 & \lambda^m \end{bmatrix}, \text{ by (i)}$$

$$= \begin{bmatrix} \lambda^{m+1} & (m+1)\lambda^m \\ 0 & \lambda^{m+1} \end{bmatrix}$$

Thus if the result is true for m, it is true for $m+1$ and as it is true for $m = 1, 2$, it is ture for any positive integer. Hence

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$$

EXAMPLES - A

1. Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 0 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 0 & 1 \end{bmatrix}$$

Show that

(i) $(AB)\,C = A\,(BC)$ (ii) $A\,(B+C) = AB + AC$

2. Show that $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$ and $\begin{bmatrix} -2 & -1 & -6 \\ 3 & 2 & 9 \\ -1 & -1 & -4 \end{bmatrix}$ commute.

3. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ Prove that $A^3 = A^{-1}$

4. Find all possible products of the matrices

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 6 \\ 7 \end{bmatrix}.$$

5. Determine the matrix M Such that $AMB = C$

where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$

9. If $A(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$,

prove (i) $A(\alpha).A(\beta) = A(\alpha+\beta)$

(ii) $A^m(\theta) = A(m\theta)$

7. Given $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

show that

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

$$B^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{1}{2}n(n-1)\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$$

1. If A, B are symmetric matrices, show that

(i) $A+B$ is symmetric

(ii) AB is symmetric if A and B commutate

(iii) A^{-1} is symmetric

(iv) adj. A is symmetric

(v) $a_0A^n + a_1A^{n-1} + \ldots + a_nI$, where $a_0, a_1 \ldots a_n$ are scalars and n is bositive integer, is symmetric.

(vi) AA^* is symmetric.

9. If m-square matrix A is symmetric and if B is of order $m \times n$, then the matrix B^*AB is symmetric.

10. Given $A = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$, show that

$$A^{4n} = I_3,\ A^{4n+1} = A,\ A^{4n+2} = -I_3 \text{ and } A^{4n+3} = -A.$$

11. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Show that (i) adj. $A = 3A^*$ (ii) adj. (adj $\cdot$ A) $= |A|\,A$

12. Obtain the inverses of following matrices :—

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & 16 \\ 1 & 0 & 3 & -5 \\ -2 & 0 & 0 & 10 \end{bmatrix}$

13. If $D = [d_i\,\delta_{ij}]$ is adiagonal matrix, prove that its inverse is given by

$$D^{-1} = \left[\frac{1}{d_i}\delta_{ij}\right].$$

14. If A, B are n-square matrices and A has inverse, show that

$$(A+B)\,A^{-1}\,(A-B) = (A-B)\,A^{-1}\,(A+B)$$

15. Find p, q, r, s, such that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

16. Show that

$$\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

17. Prove the following matrices are orthogonal and hence find their in verse :

(i) $\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

18. Show that for square matrix A :

(i) $\text{adj.}(AB) = (\text{adj.}B)(\text{adj.}A)$

(ii) $\text{adj.}(\text{adj.}A) = |A|^{n-2}A$, if $|A| \neq 0$.

(iii) $|\text{adj}(\text{adj}\,A)| = |A|^{(n-1)^2}$

(iv) $(\text{adj}\,A)^{-1} = \text{adj}\,A^{-1}$.

19. For what values of x, the matrix

$$\begin{bmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 1 \\ -2 & -4 & 1-x \end{bmatrix}$$

is singular.

20. Obtain the canonical form of matrix to be row-equivalent to each of the following matrices and reduce each to the normal form. Hence find the rank in each case

(i) $\begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix}$ (ii $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

21. Obtain the ranks of A, B, A+B and AB, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ 0 & 3 & 2 \end{bmatrix}$$

22. Discuss the consistency of the following systems of equations and solve them whenever possible.

(i) $x_1+2x_2+3x_3+x_4=0$
$x_1+x_2-x_3-x_4=0$
$3x_1-x_2+2x_3+3x_4=0$

(ii) $x_1+2x_2+2x_3=1$
$2x_1+2x_2+3x_3=3$
$x_1-x_2+3x_3=5$

(iii) $2x_1+x_2=4$
$x_1-2x_2+2x_3=7$
$3x_1+2x_2=1$

(iv) $2x_1+x_2+5x_3+x_4=5$
$x_1+x_2-3x_3-4x_4=-1$
$3x_1+6x_2-2x_3+x_4=8$
$2x_1+2x_2+2x_3-3x_4=2$

(v) $2x_1-x_2+3x_3=0$
$3x_1+2x_2+x_3=0$
$x_1-4x_2+5x_3=0$

(vi) $2x_1+x_2+5x_3=4$
$3x_1-2x_2+2x_3=2$
$5x_1-8x_2-4x_3=1$

(vii) $x_1+2x_2+x_3=-1$
$6x_1+x_2+x_3=-4$
$2x_1-3x_2-x_3=0$
$-x_1-7x_2-2x_3=7$
$x_1-x_2=1$

23. Determine the values of λ for which following sets of equations possess a non-trivial solution and obtain these solutions for the real values of λ :

(i) $3x_1+x_2-\lambda x_3=0$
$4x_1-2x_2-3x_3=0$
$2\lambda x_1+4x_2+\lambda x_3=0$

(ii) $2x_1-2x_2+x_3=\lambda x_1$
$2x_1-3x_2+2x_3=\lambda x_2$
$-x_1+2x_2=\lambda x_3$

(iii) $(1-\lambda)x_1+2x_2+3x_3=0$
$3x_1+(1-\lambda)x_2+2x_3=0$
$2x_1+3x_2+(1-\lambda)x_3=0$

24. For what values of λ the following set of equations are consistent and solve them.

$$x_1+2x_2+x_3=3,\ x_1+x_2+x_3=\lambda,\ 3x_1+x_2+3x_3=\lambda^2.$$

25. Given the following linear transformations

$$\left.\begin{aligned} x_1&=y_1-2y_2+y_3\\ x_2&=2y_1+y_2-3y_3 \end{aligned}\right\} \qquad \begin{aligned} y_1&=z_1+2z_2\\ y_2&=2z_1-z_2\\ y_3&=2z_1+3z_2 \end{aligned}$$

Find by using the method of matrices, the transformation from (z_1, z_2) to (x_1, x_2).

26. Given the linear transformation

$$[Y]=\begin{bmatrix}1&1&2\\1&2&5\\1&3&3\end{bmatrix}[X],$$

find the co-ordinates (x_1, x_2, x_3) corresponding to $(2, 0, 5)$ in Y.

27. If P be a symmetric matrix and Q be a skew symmetric matrix, show that (i) QPQ^T is symmetric
(ii) Q^3 is skew symmetric

28. It the matrix $A = \begin{bmatrix} -4 & -3 & -2 \\ -1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ show that A is a singular matrix and that adj. A is symmetric.

29. Find whether following vectors are lineary independent or dependent set and if dependent find the relation between them.

(i) $(1, 2, 3,)^T$, $(3, -2, 9)^T$, $(1, -6, -5)^T$

(ii) $x_1 = (2, 3, 4, -2)^T$, $x_2 = (-1, -2, -2, 1)^T$, $x_3 = (1, 1, 2, -1)^T$

(iii) $x_1 = [o, a-b, a-c, b+c]^T$, $x_2 = [b-a, o, b-c, c+a]^T$

$x_3 = [c-a, c-b, o, a+b]^T$, $x_4 = [b+c, c+a, a+b, o]^T$

ANSWERS

(4) $\frac{1}{6}\begin{bmatrix} 2 & -2 \\ 4 & 1 \\ 1 & 1 \end{bmatrix}$

(12) (i) $\frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$ (ii) $\frac{1}{6}\begin{bmatrix} 5 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

(15) $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

(19) $x = 1, 2, 3$

(20) (i) $\begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -\frac{3}{2} & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

(21) (i) Rank = 2 (ii) Rank = 2
(iii) Rank = 2 (iv) Rank = 3

(22) (i) $x_1 = -\frac{1}{3}c$, $x_2 = \frac{2}{3}c$, $x_3 = -\frac{2}{3}c$, $x_4 = c$

(ii) $x_1 = 1$, $x_2 = -1$, $x_3 = 1$

(iii) $x_1 = 2 - \frac{c}{2}$, $x_2 = -\frac{5}{2} + \frac{3}{4}c$, $x_3 = c$

(iv) $x_1 = 2$, $x_2 = \frac{1}{5}$, $x_3 = 0$, $x_4 = \frac{4}{5}$

(v) $x_1 = -x_2 = -x_3 = c$ (vi) $x_1 = -1$, $x_2 = -2$, $x_3 = 4$,
(vi) No solution exists. sol. is unique.

(23) (i) $\lambda = -9,\ x_1 = -\dfrac{3}{2}c,\ x_2 = -\dfrac{9}{2}c,\ x_3 = c$

$\lambda = 1,\ x_1 = \dfrac{1}{2}c,\ x_2 = -\dfrac{1}{2}c,\ x_3 = c$

(ii) $\lambda = 1,\ x_1 = 2c_1 - c_2,\ x_2 = c_1,\ x_3 = c_2$

$\lambda = -3,\ x_1 = c_1,\ x_2 = 2c_1,\ x_3 = -c_1$

(iii) $\lambda = 6;\ x = y = z = c$

(24) $\lambda = 2;\ x_1 = c,\ x_2 = -1,\ x_3 = 1 - c$

$\lambda = 3;\ x_1 = c,\ x_2 = 0,\ x_3 = 3 - c.$

(25) $x_1 = -z_1 + 7z_2;\ x_2 = -2z_1 - 6z_2$

(26) $\left(\dfrac{13}{5}, \dfrac{11}{5}, \dfrac{7}{5}\right)$

(27) (i) Independent (ii) dependent, $\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{X}_3 = 0$

(iii) Dependent, $(a+b)\mathbf{X}_1 - (a+b)\mathbf{X}_2 + (a-b)\mathbf{X}_3 - (a-b)\mathbf{X}_4 = 0.$

1·24 Characteristic roots and characteristic vectors :

Consider the linear transformation $\mathbf{Y} = \mathbf{AX}$. This transforms column vector

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ into column vector } \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and the transformation *square* matrix is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \cdots & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$

It is interesting to consider the transformation for a given matrix $\mathbf{A}$ which gives $\mathbf{Y} = \lambda\mathbf{X}$ i. e. vector $\mathbf{X}$ is transformed into a vector $\mathbf{Y}$ which has the same direction as $\mathbf{X}$ but has different magnitude. For this transformation we have

$$\lambda\mathbf{X} = \mathbf{AX} \quad \text{or} \quad [\mathbf{A} - \lambda\mathbf{I}]\,\mathbf{X} = \mathbf{O} \quad \ldots \quad \ldots \quad \ldots \quad (\text{i})$$

which represents the following system of homogeneous equations in n unknowns $x_1, x_2, \ldots x_n$ (i. e. column vector $\mathbf{X}$) :-

$$\left.\begin{array}{l} (a_{11} - \lambda)\,x_1 + a_{12}\,x_2 + \ \ldots \ \ldots \ \ldots + a_{1n}\,x_n = 0 \\ a_{21}\,x_1 + (a_{22} - \lambda)x_2 + \ \ldots \ \ldots \ \ldots + a_{2n}\,x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1}\,x_1 + a_{n2}\,x_2 + \ \ldots \ \ldots \ \ldots + (a_{nn} - \lambda)\,x_n = 0 \end{array}\right\} \ldots \ \ldots \ (\text{ii})$$

This system of equations has a *trivial* solution $\mathbf{X} = 0$ for any scalar λ. If the system of homogeneous equations is to have non-zero ($X \neq 0$) solution, then coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ of the system (iii) must have a rank less than n i. e.

$$\boxed{|\mathbf{A} - \lambda\mathbf{I}| = 0} \quad \ldots \quad \ldots \quad (46)$$

when the equation (46) is expanded, we get an equation of n^{th} degree in λ and has n roots $\lambda_1, \lambda_2, \ldots \lambda_n$ which are called *characteristic values* or *eigen values* (or latent roots of matrix $\mathbf{A}$). Corresponding to n characteristic roots, we get n values of column vector $\mathbf{X}$, obtained by solving equations (ii) or

$$\boxed{\mathbf{AX} = \lambda\mathbf{X}} \quad \ldots \quad \ldots \quad \ldots \quad (47)$$

These column vectors are known as *characteristic vectors* or *eigen vectors* of matrix $\mathbf{A}$.

For matrix $\mathbf{A}$, the polynomial

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

is known as *characteristic polynomial* of matrix $\mathbf{A}$.

Note :- The simplification of the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ in the characteristic polynomial is obtained directly by the method illustrated for matrix A of order 3×3.

$$\text{If } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (iv)$$

then the characteristic polynomial $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is given by

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

which can be written on simplification as

$$\lambda^3 - [\text{ sum of minors of order } \textit{one} \text{ along diagonal of } \mathbf{A}]\lambda^2$$
$$+ [\text{ sum of minors of order } \textit{two} \text{ along diagonal of } \mathbf{A}]\lambda$$
$$- |\mathbf{A}| = 0 \quad \ldots \quad \ldots \quad \ldots \quad (v)$$

$$\text{i.e.}\quad \lambda^3-\Big[a_{11}+a_{22}+a_{33}\Big]\lambda^2+\left[\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}+\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}+\begin{vmatrix} a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}\right]\lambda$$

$$-\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=0 \qquad \ldots \qquad \ldots \qquad (\text{v})$$

Ex. Find the characteristic values and characteristic vectors of matrix

$$A=\begin{bmatrix} 2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$$

The characteristic matrix is

$$[A-\lambda I]=\begin{bmatrix} 2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix} 2-\lambda & -2 & 3\\ 1 & 1-\lambda & 1\\ 1 & 3 & -1-\lambda\end{bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -2 & 3\\ 1 & 1-\lambda & 1\\ 1 & 3 & -1-\lambda\end{vmatrix}=0$$

which can be simplified by using result (iii).

Thus

$$\lambda^3-[2+1-1]\lambda^2-\left[\begin{vmatrix} 2 & -1\\ 1 & 1\end{vmatrix}+\begin{vmatrix} 1 & 1\\ 3 & -1\end{vmatrix}+\begin{vmatrix} 2 & 3\\ 1 & -1\end{vmatrix}\right]\lambda$$

$$+\begin{bmatrix} 2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}=0$$

$$\text{i.e.}\quad \lambda^3-2\lambda^2+[4-4-5]\lambda-(-6)=0$$

$$\text{i.e.}\quad \lambda^3-2\lambda^2-5\lambda+6=0$$

On factorisation, we get

$$(\lambda+2)(\lambda-1)(\lambda-3)=0$$

$\therefore$ The characteristic values of matrix **A** are

$$\lambda_1=-2,\ \lambda_2=1,\ \lambda_3=3.$$

The characteristic vector **X** has to satisfy the equation

$$[A-\lambda I]X=0$$

which for the given matrix **A** is

$$\begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots \ldots (i)$$

Corresponding to $\lambda_1 = -2$, the characteristic vector X_1 is given by [from (i)]

$$\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x^3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i. e.
$$\left.\begin{aligned} 4x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Solving first two equations of (ii), we get proportional values of x_1, x_2, x_3 i.e.

$$\frac{x_1}{-2-9} = \frac{x_2}{3-4} = \frac{x_3}{12+2} \text{ i. e. } \frac{x_1}{11} = \frac{x_2}{1} = \frac{x_3}{19} = k$$

For $k = 1$, the eigen vector corresponding to $\lambda_1 = -2$ is

$$X_1 = [\,11, 1, -14\,]^T.$$

For $\lambda = 1$, the equations [from (i)] are

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 + 0x_2 + x_3 &= 0 \\ x_1 + 3x_2 - 2x_3 &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (iii)$$

Solving first two equations for proportional values of x_1, x_2, x_3 we get

$$\frac{x_1}{-2-0} = \frac{x_2}{3-1} = \frac{x_3}{0-(-2)} \text{ i e. } \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-1} = k.$$

The characteristic vector x_2 corresponding to $\lambda_2 = 1$

is $X_2 = [\,1, -1, -1\,]^T$ where $k = 1$.

Similarly from (i) for $\lambda = 3$, the set of equations are

$$x_1 - 2x_2 + 3x_3 = 0.\ x_1 - 2x_2 + x_3 = 0,\ x_1 - 3x_2 - 4x_3 = 0.$$

Solving first two of the equations for proportional values of x_1, x_2, x_3 we get

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}, \text{ i. e. } \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k.$$

Thus for $k = 1$, the eigen vector X_3 for $\lambda_3 = 3$ is given by

$$X_3 = [\,1, 1, 1\,]^T$$

Thus the three eigen vectors or characteristic vectors are

$$X_1 = \begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix}_{\lambda=-2}, \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}_{\lambda=1}, \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\lambda=3}$$

1·25 Properties of characteristic vectors :–

If λ_r and λ_s are two distinct characteristic values of an n–square matrix **A**, having characteristic vectors $\mathbf{X}_r$ and $\mathbf{X}_s$ respectively then

(I) $\begin{cases} \text{(a)} & \mathbf{X}_r \text{ and } \mathbf{X}_s \text{ are always linearly independent} \\ \text{(b)} & \mathbf{X}_r \text{ and } \mathbf{X}_s \text{ are orthogonal, if } A \text{ is symmetric.} \end{cases}$

(a) Since $\mathbf{X}_r$ and $\mathbf{X}_s$ are characteristic vectors corresponding to values λ_r and λ_s, we have

$$\mathbf{AX}_r = \lambda_r \mathbf{X}_r \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\mathbf{AX}_s = \lambda_s \mathbf{X}_s \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Let c_r and c_s be two scalars (numbers) such that

$$c_r \mathbf{X}_r + c_s \mathbf{X}_s = 0 \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

By art. 1·22, the vectors $\mathbf{X}_r$ and $\mathbf{X}_s$ will be linearly independent, if we show that whenever (iii) is true, $c_r = c_s = 0$.

Premultiplying (iii) by **A**, we get

$$c_r\, \mathbf{AX}_r + c_s\, \mathbf{AX}_s = 0$$

using (i) and (ii), we have

$$c_r \lambda_r \mathbf{X}_r + c_s \lambda_s \mathbf{X}_s = 0 \ \ldots \quad \ldots \quad \text{(iv)}$$

Multiplying (iii) by λ_r and subtracting from (iv), we get

$$c_s (\lambda_s - \lambda_r) \mathbf{X}_s = 0$$

Since $\lambda_r \neq \lambda_s$ and $\mathbf{X}_s$ is non-zero vector, it follows that $c_s = 0$. Hence from (iii) as $\mathbf{X}_s$ is non-zero vectors $c_r = 0$. As $c_r = c_s = 0$, we have $\mathbf{X}_r$ and $\mathbf{X}_s$ as linearly independent vectors.

(b) Since **A** is symmetric $\mathbf{A} = \mathbf{A}^T$.

Now we have

$$\mathbf{AX}_r = \lambda_r \mathbf{X}_r \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\mathbf{AX}_s = \lambda_s \mathbf{X}_s \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Taking transpose of (ii), we get

$$\mathbf{X}_s^T \mathbf{A}^T = \lambda_s \mathbf{X}_s^T$$

i.e. $\mathbf{X}_s^T \mathbf{A} = \lambda_s \mathbf{X}_s^T \quad (\text{as } \mathbf{A} = \mathbf{A}^T)$

Post multiply by vector $\mathbf{X}_r$, we have

$$\mathbf{X}_s^T \mathbf{A}\mathbf{X}_r = \lambda_s \mathbf{X}_s^T \mathbf{X}_r$$

i. e. $\mathbf{X}_s^T \lambda_r \mathbf{X}_r = \lambda_s \mathbf{X}_s^T \mathbf{X}_r$

i. e. $(\lambda_r - \lambda_s)\mathbf{X}_s^T \mathbf{X}_r = 0$

as $\lambda_r \neq \lambda_s, \mathbf{X}_s^T \mathbf{X}_r = 0$

Hence $\mathbf{X}_r$ and $\mathbf{X}_s$ are orthogonal.

(**b**) **The characteristic vectors corresponding to distinct characteristic values are independent vectors** :—

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_k$ be characteristic vectors corresponding to distinct characteristic values $\lambda_1, \lambda_2 \ldots \lambda_k$ of matrix **A.**

Let $\mathbf{X}_1, \mathbf{X}_2 \ldots \mathbf{X}_r$ $(r < k)$ be independent set of vectors.

Now let $\mathbf{X}_{r+1}$ be a vector dependent on $\mathbf{X}_1, \mathbf{X}_2 \ldots \mathbf{X}_r$. Thus the relation

$$c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 \cdots + c_r \mathbf{X}_r + c_{r+1} \mathbf{X}_{r+1} = 0 \quad \ldots \quad \ldots \quad (\text{i})$$

implies that not all of $c_1, c_2 \ldots c_{r+1}$ are zero.

Premultiply the equation (i) by matrix **A**, we get

$$c_1 [\mathbf{A}\mathbf{X}_1] + c_2 [\mathbf{A}\mathbf{X}_2] + \ldots + c_r [\mathbf{A}\mathbf{X}_r] + c_{r+1} [\mathbf{A}\mathbf{X}_{r+1}] = 0$$

i. e. $c_1 \lambda_1 \mathbf{X}_1 + c_2 \lambda_2 \mathbf{X}_2 + \ldots + c_r \lambda_r \mathbf{X}_r + c_{r+1} \lambda_{r+1} \mathbf{X}_{r+1} = 0 \ldots$ (ii)

[as $\mathbf{A}\mathbf{X}_i = \lambda_i \mathbf{X}_i$]

Multiply (i) by λ_{r+1} and subtract from (ii), we get

$$c_1 (\lambda_1 - \lambda_{r+1}) \mathbf{X}_1 + c_2 (\lambda_2 - \lambda_{r+1}) \mathbf{X}_2 + \ldots + c_r (\lambda_r - \lambda_{r+1}) \mathbf{X}_r = 0$$

Since characteristic values are distinct and $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_r$ are independent, by are 2·2, we have

$$c_1 = c_2 = \ldots \ldots = c_r = 0$$

and from (ii), we get

$$c_{r+1} \lambda_{r+1} \mathbf{X}_{r+1} = 0 \quad \text{i. e.} \quad c_{r+1} = 0$$

Since $c_1, c_2, \ldots c_{r+1}$ are all zero, the relation (i) contradicts the statement $\mathbf{X}_{r+1}$ is dependent on $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_r$. Hence $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_{r+1}$ are independent characteristic vectors. Taking $\mathbf{X}_{r+2}$ as dependent on $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_{r+1}$, we can similarly show that $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_{r+1}, \mathbf{X}_{r+2}$ are independent vectors. Continuing the process upto $\mathbf{X}_m$, it can be shown the characteristic vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_m$ corresponding to m distinct characteristic values are independent.

For a matrix of order 3,3, the three characteristic vectors are always independent vectors and hence we come across following cases for determination of 3 independent characteristic vectors of matrix **A** of order 3×3 :—

(i) If the three characteristic values $\lambda_1, \lambda_2, \lambda_3$ are distinct for a matrix $\mathbf{A}_{3\times3}$, then the corresponding characteristic vectors $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are independent by property (a) of art. 1·25.

(ii) If two of the eigen values of $\mathbf{A}_{3\times3}$ are equal ($\lambda_1 = \lambda_2$) then the corresponding *independent* characteristic vectors are obtaind as follows :—

(a) If A is symmetric matrix, we determine the vector $\mathbf{X}_1$ to satisfy the equation $\mathbf{AX}_1 = \lambda_1\mathbf{X}_1$ and vector $\mathbf{X}_2$ corresponding to $\lambda_2 = \lambda_1$ is determined as orthogonal to $\mathbf{X}_1$ and $\mathbf{X}_3$ [by property (b)].

(b) If A is non-symmetric matrix, then $\mathbf{X}_1, \mathbf{X}_2$ are obtained from two independent solutions of $\mathbf{AX} = \lambda_1 \mathbf{X}$.

This is illustrated in the following example :

Ex. 1. Find eigen values and eigen vectors of the following matrices.

(a) $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(a) This example with three distinct eigen values, is solved on pp. 79.

(b) The charecteristic equation of matrix A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

i. e. $\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$

which gives $(\lambda-1)(\lambda-1)(\lambda-7) = 0$

Therefore eigen values are $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 7$

Here eigen values $\lambda_1 = \lambda_2$.

The eigen vectors are given by $[A - \lambda I] X = 0$ i. e.

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots \ldots \text{(ii)}$$

For non-repeated eigen value $\lambda_3=7$, the equation (ii) are given by

$$-5x_1+x_2+x_3=0,\ 2x_1-4x_2+2x_3=0,\ 3x_1+3x_2-3x_3=0$$

Solving first two equations for proportional valves of x_1, x_2, x_3, we get

$$\frac{x_1}{6}=\frac{x_2}{12}=\frac{x_3}{18}=k \quad \text{i. e.} \quad x_3=[\,1,2,3\,]^T$$

For the other two equal eigen values $\lambda_1=\lambda_2=1$, the equation (ii), gives only one equation

$$x_1+x_2+x_3=0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Since the matrix is *non-symmetric* and hence the two *linearly independent* eigen vectors x_1, x_2 are respectively given by $x_1=0$ and $x_2=0$

i. e. $x_1=0, x_2=1, x_3=-1$ thus $x_1=[\,0,1,-1\,]^T$

and when $x_2=0$,

$x_1=1, x_2=0, x_3=-1$, thus $x_2=[\,1,0,-1\,]^T$

Thus the eigen vectors of matrix are

$$\mathbf{X}_1=\begin{bmatrix}0\\1\\-1\end{bmatrix}_{\lambda=1},\ \mathbf{X}_2=\begin{bmatrix}1\\0\\-1\end{bmatrix}_{\lambda=1},\ \mathbf{X}_3=\begin{bmatrix}1\\2\\3\end{bmatrix}_{\lambda=7}$$

we can show that $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are linearly independent vectors.

(c) For the matrix the characteristic equation is

$$\begin{vmatrix}6-\lambda & -2 & 2\\-2 & 3-\lambda & -1\\2 & -1 & 3-\lambda\end{vmatrix}=0$$

i. e. $\lambda^3-12\lambda^2+36\lambda-32=0$

i. e. $(\lambda-2)(\lambda-2)(\lambda-8)=0$

thus the eigen values are $\lambda_1=\lambda_2=2, \lambda_3=8$

The eigen vectors are given by $[A-\lambda I]X=0$ i. e.

$$\begin{bmatrix}6-\lambda & -2 & 2\\-2 & 3-\lambda & -1\\2 & -1 & 3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \ldots \qquad \ldots \qquad \text{(iv)}$$

For non repeated eigen value $x_3=8$, the equation (iv) gives

$$-2x_1-2x_2+2x_3=0,\ -2x_1-5x_2-x_3=0,\ 2x_1-x_2-5x_3=0$$

solving first two equations for proportional values of x_1, x_2, x_3, we get

$$\frac{x_1}{12}=\frac{x_2}{-6}=\frac{x_3}{6}=k \quad \text{i.e.} \quad x_3=[\,2,-1,1\,]^T$$

For repeated eigen values $\lambda_1=\lambda_2=2$, the equation (iv) gives only one equation

$$2x_1-x_2+x_3=0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(v)}$$

To determine the eigen vector $\mathbf{X}_1$, we take $x_1 = 0$, $x_2 = 1$, $x_3 = 1$ to satisfy equation (v) i.e.

$$\mathbf{X}_1 = [0, 1, 1]^T$$

Since the matrix is symmetric, the eigen vector $\mathbf{X}_3 = [l, m, n]^T$ is so chosen that it is orthogonal to $\mathbf{X}_2$ and $\mathbf{X}_1$ i.e.

$$\mathbf{X}_2^T \mathbf{X}_3 = 0, [2 \ -1 \ 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \text{ i.e. } 2l - m + n = 0 \quad \ldots\ldots (vi)$$

$$\text{and } \mathbf{X}_1^T \mathbf{X}_3 = 0, \ [0, 1, 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \text{ i.e. } 0l + m + n = 0 \quad \ldots (vii)$$

solving (vi) and (vii) for proportional values of l, m, n we get

$$\frac{l}{-2} = \frac{m}{-2} = \frac{n}{2} = k \text{ i.e. } \mathbf{X}_3 = [1, 1, -1]^T$$

Thus the eigen vecters of the symmetric matrix A are

$$\mathbf{X}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 2$,

It can be shown that the eigen vectors $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ of symmetric matrix A are orthogonal to each other and also $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are linearly independent.

1·26 Caley-Hamilton Theorem :—

A matrix satisfies its own characteristic equation :—

For a square matrix of order n, if the characteristic equation is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0, \ (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n] = 0$$

then

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} \ldots\ldots + a_n \mathbf{I} = \mathbf{O}$$

We have from the result (31) of art. 1·15,

$$[\mathbf{A} - \lambda \mathbf{I}] \text{ adj } [\mathbf{A} - \lambda \mathbf{I}] = |\mathbf{A} - \lambda \mathbf{I}| \, \mathbf{I} \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

Since adj $[\mathbf{A} - \lambda \mathbf{I}]$ has elements as cofactors of elements of $|\mathbf{A} - \lambda \mathbf{I}|$, the elements of adj $[\mathbf{A} - \lambda \mathbf{I}]$ are polynomials in λ of degree $n - 1$ or less, as the elements of $[\mathbf{A} - \lambda \mathbf{I}]$ are at most of the first degree in λ.

Hence adj $[\mathbf{A}-\lambda\mathbf{I}]$ can be written as a matrix polynomial in λ given by

$$\text{adj}\,[\mathbf{A}-\lambda\mathbf{I}] = \mathbf{B}_0\lambda^{n-1}+\mathbf{B}_1\lambda^{n-2}+\ldots\mathbf{B}_{n-2}\lambda+\mathbf{B}_{n-1} \quad \ldots \text{(ii)}$$

Thus from (i) and (ii), we get

$$|\mathbf{A}-\lambda\mathbf{I}|\;\mathbf{I} = [\mathbf{A}-\lambda\mathbf{I}]\;\text{adj}\,[\mathbf{A}-\lambda\mathbf{I}]$$

i. e. $(-1)^n\,[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2}\ldots + a_n\mathbf{I}]$

$$= [\mathbf{A}-\lambda\mathbf{I}]\,[\mathbf{B}_0\lambda^{n-1}+\mathbf{B}_1\lambda^{n-2}\ldots+\mathbf{B}_{n-2}\lambda+\mathbf{B}_{n-1}] \ldots \text{(iii)}$$

Comparing coefficients of same powers of λ on both sides, we have

$$(-1)^n\,\mathbf{I} = -\mathbf{B}_0\mathbf{I}$$
$$(-1)^n\,a_1\mathbf{I} = \mathbf{A}\mathbf{B}_0 - \mathbf{I}\,\mathbf{B}_1$$
$$(-1)^n\,a_2\mathbf{I} = \mathbf{A}\mathbf{B}_1 - \mathbf{I}\,\mathbf{B}_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(-1)^n\,a_n\mathbf{I} = \mathbf{A}\mathbf{B}_{n-1}$$

Premultiplying the above equations successively by $\mathbf{A}^n$, $\mathbf{A}^{n-1}$, $\mathbf{A}^{n-2}$ ..., $\mathbf{I}$ and adding, we get

$$(-1)^n\left\{\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \ldots a_n\mathbf{I}\right\} = \mathbf{O}$$

Thus

$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \ldots + a_n\mathbf{I} = \mathbf{O} \quad \ldots \quad \ldots \quad \text{(iv)}$$

Cor:— If $\mathbf{A}$ is non-singular matrix, the premultiplying (iv) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + \ldots + a_{n-1}\mathbf{I} + a_n\mathbf{A}^{-1} = \mathbf{O}$$

or $$\mathbf{A}^{-1} = -\frac{1}{a_n}\,[\mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + \ldots + a_{n-1}\mathbf{I}] \quad \ldots \quad \text{(v)}$$

Ex. 1 Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

satisfies its characteristic equation and hence determine $\mathbf{A}^{-1}$.

The characteristic equation of matrix $\mathbf{A}$ is

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

i.e. $$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \ldots \ldots \ldots \text{(i)}$$

Now

$$A^2 = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix}$$

$$A^3 = A\,A^2 = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} = \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -2 \\ 33 & -52 & 13 \end{bmatrix}$$

Now in L.H.S. of, (i) we replaee λ by A, we have

$$A^3 - 6A^2 + 11A - 6I$$

$$= \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -2 \\ 33 & -52 & 13 \end{bmatrix} - \begin{bmatrix} 156 & -192 & -12 \\ 84 & -90 & -24 \\ 66 & -96 & 18 \end{bmatrix} + \begin{bmatrix} 88 & -88 & -22 \\ 44 & -33 & -22 \\ 33 & -44 & 11 \end{bmatrix}$$

$$- \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad A^3 - 6A^2 + 11A - 6I = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Thus A satisfy its characteristic equaction (i). To find A^{-1}, premultiply (ii) by A^{-1}, we get

$$A^2 - 6A + 11I - 6A^{-1} = 0$$

i.e. $\quad 6A^{-1} = A^2 - 6A + 11I$

$$= \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} - \begin{bmatrix} 48 & -48 & -12 \\ 24 & -18 & -12 \\ 18 & -24 & 6 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 16 & 10 \\ -10 & 14 & 8 \\ -7 & 8 & 8 \end{bmatrix}$$

Hence $\quad A^{-1} = \dfrac{1}{6}\begin{bmatrix} -11 & 16 & 10 \\ -10 & 14 & 8 \\ -7 & 8 & 8 \end{bmatrix}$

Ex. 2 Find the characteristic equaction for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and use it find the simplified expression for

$$A^6 + 3A^4 - 6A^3 + 2A^2 - \text{[illegible]}$$

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

i. e. $\quad \lambda^2 - 4\lambda - 5 = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (i)$

Divide the polynomial $\lambda^5 + 5\lambda^4 - 6\lambda^3 + 2\lambda^2 - 4\lambda + \lambda$ corresponding to given matrix expression, by $\lambda^2 - 4\lambda - 5$. We get the quotient $\lambda^3 + 9\lambda^2 + 35\lambda + 147$ and remainder is $759\lambda + 742$. Thus

$$\lambda^5 + 5\lambda^4 - 6\lambda^3 + 2\lambda^2 - 4\lambda + 7 = (\lambda^2 - 4\lambda - 5)(\lambda^3 + 9\lambda^2 + 35\lambda + 147) + 759\lambda + 742$$

Replacing λ by matrix A, we get

$$A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$$
$$= (A^2 - 4A - 5I)(A^3 + 9A^2 + 35A + 147I) + 759A + 742I \quad \ldots \ldots \ldots \quad (ii)$$

Since A satisfies its characteristic equation. $\lambda^2 - 4\lambda - 5 = 0$

$$\therefore \quad A^2 - 4A - 5I = 0$$

Substituting in (ii), we get

$$A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I = 759A + 742I$$
$$= 759\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 742\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1501 & 3036 \\ 1518 & 3019 \end{bmatrix}$$

1·27 Similarity of Matrices :–

If A, B are two square matrices of oreder n, then B is said to be similar to A, if there exists **non-singular** matrix P such that

$$\boxed{B = P^{-1}AP} \quad \ldots \ldots \ldots \ldots \ldots \quad (49)$$

For similar matrices A, B, we have

(i) $|A| = |B|$

(ii) The characteristic polynomials for A and B are same.

Since A, B are similar, we have

$$B = P^{-1}AP \qquad [\text{ by } (49)]$$

(i) Taking determinant of both the sides, we get

$$|B| = |P^{-1}AP| = |P^{-1}|\ |A|\ |P|$$
$$= |P^{-1}|\ |P|\ |A| \quad [\text{ being determinants we commutate}]$$
$$= |P^{-1}P|\ |A|$$
$$= |I|\ |A| = |A| \qquad [\text{ as } |I| = 1]$$

(ii) Consider the determinant of matrix $B - \lambda I$ i. e.

$$
\begin{aligned}
|B-\lambda I| &= |P^{-1}AP - \lambda I| \qquad [\text{ from } (49)\,] \\
&= |P^{-1}AP - \lambda P^{-1}IP| \\
&= |P^{-1}(A-\lambda I)\,P| \\
&= |P^{-1}|\;|A-\lambda I|\;|P| \\
&= |P^{-1}|\;|P|\;|A-\lambda I| \qquad [\text{ being determinants, they commute}\,] \\
&= |P^{-1}P|\;|A-\lambda I| \\
&= |I|\;|A-\lambda I| \\
&= |A-\lambda I|
\end{aligned}
$$

Since $|B-\lambda I| = |A-\lambda I|$ the similar matrices A, B have same characteristic equction.

Note :– When A, B are to be proved similar, show that $|A-\lambda I| = |B-\lambda I|$

1·28 Reduction of matrix to a dialogonal matrix which has elements as characteristic values :—

Let $X_1 = [\,l_1, m_1, n_1\,]^T$, $X_2 = [\,l_2, m_2, n_2\,]^T$, $X_3 = [\,l_3, m_3, n_3\,]^T$ be the characteristic value of matrix $A_{3\times 3}$, corresponding to the characteristic values $\lambda_1, \lambda_2, \lambda_3$. Thus

$$
AX_1 = \lambda_1 X_1,\quad AX_2 = \lambda_2 X_2,\quad AX_3 = \lambda_3 X_3
$$

i. e.
$$
A\begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix},\quad A\begin{bmatrix} l_2 \\ m_2 \\ n_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} l_2 \\ m_2 \\ n_2 \end{bmatrix},
$$
$$
A\begin{bmatrix} l_3 \\ m_3 \\ n_3 \end{bmatrix} = \lambda_3 \begin{bmatrix} l_3 \\ m_3 \\ n_3 \end{bmatrix}
$$

The above three results can be expressed as

$$
A\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \ldots\ldots(i)
$$

The matrix

$$
L = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}
$$

formed by column characteristic vectors of matrix A, is known as **modal matrix.**

and the diagonal matrix.

$$
\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
$$

is known as λ matrix.

Thus the result (i) can be expressed as

$$AL = L\Lambda$$

Pre multiplying by L^{-1}, we get

$$\boxed{L^{-1}AL = \Lambda} \quad \ldots \ldots \ldots \ldots \ldots \quad (50)$$

Thus the matrix A is similar to the diagonal matrix Λ.

Ex. 1 Find the matrix **P** such that $P^{-1}AP$ is diagonal matrix, where

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

From the example of art. 1.24, the characteristic vectors of the matrix A are

$$X_1 = \begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix}_{\lambda = -2}, \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\lambda = 1} \text{ and } X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\lambda = 3}$$

Then the matrix **P** is formed by these column vectors i. e.

$$P = \begin{bmatrix} 11 & 1 & 1 \\ 1 & -1 & 1 \\ -14 & -1 & 1 \end{bmatrix} \quad \ldots \ldots \ldots (i)$$

$$\text{Now } P^{-1} = -\frac{1}{30}\begin{bmatrix} 0 & -2 & 2 \\ -15 & 25 & -10 \\ -15 & -3 & -12 \end{bmatrix}$$

Hence

$$P^{-1}AP = -\frac{1}{30}\begin{bmatrix} 0 & -2 & 2 \\ -15 & 25 & -10 \\ -15 & -3 & -12 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 11 & 1 & 1 \\ 1 & -1 & 1 \\ -14 & -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{30}\begin{bmatrix} 0 & -2 & 2 \\ -15 & 25 & -10 \\ -15 & -3 & -12 \end{bmatrix}\begin{bmatrix} -22 & 1 & 3 \\ -2 & -1 & 3 \\ 28 & -1 & 3 \end{bmatrix}$$

$$= -\frac{1}{30}\begin{bmatrix} 60 & 0 & 0 \\ 0 & -30 & 0 \\ 0 & 0 & -90 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Thus the matrix **P** is given by (i)

Ex. 2 Show that **AB** and **BA** have same characteristic equation, if **A**, **B** are symmetric matrices.

Consider the matrix **C** such that

$$C = AB - \lambda I \quad \ldots \ldots \ldots \ldots \quad (i)$$

Now

$$C^T = [AB - \lambda I]^T = B^T A^T - \lambda I$$

$$= BA - \lambda I \quad .. \quad .. \quad .. \quad .. \quad .. \quad \text{(ii)}$$

$$\text{as } A = A^T,\ B = B^T$$

Since $|C| = |C^T|$, we have from (i) and (ii)

$$|AB - \lambda I| = |BA - \lambda I|$$

Hence, **AB** and **BA** have same characteristic equations.

Ex. 3. If λ is an eigen value of matrix **A**, show that

(a) λ^n is eigen value of A^n

(b) $\dfrac{|A|}{\lambda}$ is eigen value of adj **A**.

(a) Since λ is an eigen value of **A**, we have

$$AX = \lambda X \quad .. \quad .. \quad .. \quad .. \quad ..\text{(i)}$$

Premultiply by **A**, we get

$$A^2X = \lambda AX = \lambda(\lambda X) \qquad [\text{from (i)}]$$

$$= \lambda^2 X \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad \text{(ii)}$$

Premultiplying (ii) by **A**, we get

$$A^3X = \lambda^2(AX) = \lambda^3 X \quad .. \quad .. \quad .. \quad .. \quad .. \quad ..\text{(iii)}$$

Thus premultiply (i) by A^{n-1}, we have

$$A^nX = \lambda^n X$$

which shows that λ^n is an eigen value of A^n

(b) Now

$$A.\text{adj } A = |A|\, I$$

or $$\text{adj } A = |A|\, A^{-1} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad \text{(i)}$$

As λ is an eigen value of **A**, we have

$$AX = \lambda X$$

Pre multiplying by A^{-1}, we get

$$A^{-1}(AX) = \lambda A^{-1}X$$

$$\therefore \quad X = \lambda A^{-1}X$$

i.e. $$A^{-1}X = \frac{1}{\lambda} X \quad .. \quad .. \quad .. \quad ..\text{(ii)}$$

i. e. $\dfrac{1}{\lambda}$ is eigen value of A^{-1}

Now post-multiply by **X** the result (i)

i. e.

$$(\text{adj } \mathbf{A})\ \mathbf{X} = |\mathbf{A}|\ \mathbf{A}^{-1}\mathbf{X}$$

$$= |\mathbf{A}|\ \frac{1}{\lambda}\mathbf{X} \quad [\text{ by result (ii) }]$$

Hence

$$(\text{adj } \mathbf{A})\ \mathbf{X} = \frac{|\mathbf{A}|}{\lambda}\ \mathbf{X}$$

Thus $\dfrac{|\mathbf{A}|}{\lambda}$ is the eigen value of adj **A**

EXAMPLE-B

1. (a) Show that **A**, $\mathbf{A}^T$ has same characteristic equation

   (b) If **A** is real, symmetric and orthogonal, prove that its eigen values are ± 1

   (c) Prove that $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{BA}^{-1}$ have same characteristic equation

2. If λ is an eigen value of n-square matrix **A**, show that

   (i) λ^{-1} is an eigen value of $\mathbf{A}^{-1}$

   (ii) $K\lambda$ is an eigen value of $K\mathbf{A}$

   (iii) $K+\lambda$ is an eigen value of $\mathbf{A}+KI$

3. Find the characteristic values and characteristic vectors of the following matrices

(i) $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(vii) $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -1 & -1 & -8 \end{bmatrix}$ (viii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ix) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

4. Verify Caley-Hamilton theorem for the following matrices, also find $\mathbf{A}^{-1}$ in each case.

(i) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

5. For a symmetric matrix A the eigen vectors are $[1, 1, 1]^T$ $[1, -2, 1]^T$ corresponding to eigen value $\lambda_1 = 2$, and $\lambda_2 = 4$ respectively. If $\lambda_3 = 6$ find the matrix A

$$\text{Ans : } A = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

6. Show that the matrix

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

can be reduced to a diagonal form with diagonal elements as eigen values, find the diagonalising matrix P.

7. Find the characteristic equation of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and use it to express

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$$

in terms of A. [Ans. $A + 5I$]

8. Show that the following matrices are similar

$$\begin{bmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & f & h \\ f & 0 & g \\ h & g & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & g & f \\ g & 0 & h \\ f & h & 0 \end{bmatrix}$$

ANSWERS

(3) (i) $[1, 1, 1]^T_{\lambda=-1}$, $[0, 1, -1]^T_{\lambda=-1}$, $[1, 1, 2]^T_{\lambda=3}$

(ii) $[4, 3, 2]^T_{\lambda=1}$, $[3, 2, 1]^T_{\lambda=2}$, $[2, 1, 1]^T_{\lambda=3}$

(iii) $[1, -2, 3]^T_{\lambda=1}$, $[2, -2, 3]^T_{\lambda=1}$, $[2, 1, -2]^T_{\lambda=3}$

(iv) $[2, 1, 4]^T_{\lambda=1}$, $[1, 1, 2]^T_{\lambda=2}$, $[0, 1, 1]^T_{\lambda=5}$

(v) $[2, 2, 1]^T_{\lambda=-2}$, $[1, 1, 0]^T_{\lambda=1}$, $[-1, 0, 3]^T_{\lambda=1}$

(vi) $[1, 2, 3]^T_{\lambda=14}$, $[0, 3, -2]^T_{\lambda=0}$, $[-13, 2, 3]^T_{\lambda=0}$

(vii) $\begin{bmatrix} 4, & 1, & -1 \end{bmatrix}^T_{\lambda = 1}$, $\begin{bmatrix} 0, & 1, & 1 \end{bmatrix}^T_{\lambda = -1}$, $\begin{bmatrix} 1, & -2, & 2 \end{bmatrix}^T_{\lambda = -1}$

(viii) $\begin{bmatrix} 2, & 1, & 2 \end{bmatrix}^T_{\lambda = 8}$, $\begin{bmatrix} 0, & 2, & -1 \end{bmatrix}^T_{\lambda = -1}$, $\begin{bmatrix} 1, & 0, & 1 \end{bmatrix}^T_{\lambda = -1}$

(ix) $\begin{bmatrix} 1, & 2, & 2 \end{bmatrix}^T_{\lambda = 0}$, $\begin{bmatrix} -4, & -2, & 4 \end{bmatrix}^T_{\lambda = 3}$, $\begin{bmatrix} 2, & -2, & 1 \end{bmatrix}^T_{\lambda = 15}$

(x) $\begin{bmatrix} 1, & -i \end{bmatrix}^T_{\lambda = e^{i\theta}}$, $\begin{bmatrix} 1, & i \end{bmatrix}^T_{\lambda = e^{-i\theta}}$,

(4) (i) $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ (ii) $A^{-1} = -\begin{bmatrix} -1 & -\sqrt{2} & 0 \\ -\sqrt{2} & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

(iii) $A^{-1} = -\frac{1}{a^2+b^2+c^2} A$

CHAPTER 2

COMPLEX NUMBERS

The section deals with the fundamental definitions and operations involving complex numbers. Since the graphical operations of addition and subtractions of complex numbers are anologous to corresponding operations of vectors, complex numbers are frequently applied to vector problems in applied mathematics viz. distribution of alternating currents in electrical circuits, the study of forced vibrations of a dynamical system.

2·1. Complex Number :—

In solving quadratic equations, we often get a sq. root of negative number as part of the values for roots eg. if $x^2 - 2x + 2 = 0$, the roots are

$$x = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm \tfrac{1}{2}\sqrt{-4}$$

The extraction of sq. root is not possible as no real number exists which when squared will yield a negative number [eg. -4]. Hence a new type of number known as imaginary number is introduced to extract a sq. root of a real number.

eg. $\sqrt{-4} = \sqrt{4}\,\sqrt{-1} = 2\sqrt{-1}$; $\sqrt{-36} = 6\sqrt{-1}$

Thus the sq. root of any negative number can always be expressed in terms of $\sqrt{-1}$, known as imaginary unit and is usually denoted by 'i' or 'j'. The *fundamental property* of the symbol i is

$$\boxed{i^2 \text{ (or } j^2 \text{) } = -1} \qquad \ldots \qquad \ldots \qquad (1)$$

Introducing imaginary numbers the system of numbers can be diagramatically represented as follows :—

Numbers

- Real numbers
  - Rational numbers
  - Irrational numbers
- Imaginary numbers

[± Integral, ± fractional] [eg. $\sqrt[3]{5}$, $\sqrt[3]{7}$ ect.]

To make the real and imaginary numbers part of one system, a new number known as *complex number* represented by $x + iy$, is introduced.

If $\quad z = x + iy$ (complex number)

then x and y are known as *real* and *imaginary parts* of the complex number z.

2·2 Equality of complex numbers :—

If two complex numbers are equal, then their real and imginary parts are respectively equal.

Let $x_1 + iy_1 = x_2 + iy_2$

$\therefore \quad (x_1 - x_2) = i(y_2 - y_1)$

Squaring both the sides, we get

$(x_1 - x_2)^2 = -(y_2 - y_1)^2$ [as $i^2 = -1$]

$\therefore \quad (x_1 - x_2)^2 + (y_2 - y_1)^2 = 0$

$\therefore \quad x_1 = x_2$ and $y_1 = y_2$. Hence the result.

2·3. Graphical representation of complex number (Argand's Diagram) :--

The method of representing complex numbers by points in a plane is due to J. R. Argand and is usually known as Argand's diagram.

Let a line a units long represented by OA (Fig. 3) be turned about O from position OA, through 180°.

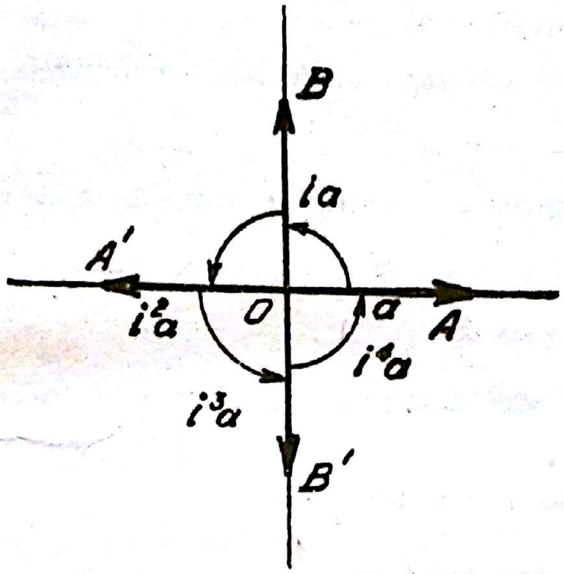

Fig. 3

It then has $-$ve direction $\overrightarrow{OA'}$ and graphically represents $-a$ units. Thereafter turning through 180°, it again takes up the position $\overrightarrow{OA}$ and graphically represents $+\ a$ units. In each instance, rotation of vector $\overrightarrow{OA}$ through 180° has been equivalent to multiplying the vector by -1 and since $i^2 = -1$, the multiplication by i may be taken as equivalent to *rotating the line* through 90°. Hence definite geometrical meaning may be attached to the complex number $x + iy$.

The complex number $x + iy$ represents in vector notation, a vector 'x' added to vector 'y' perpendicular to the vector 'x' [from the geometrical meaning of i].

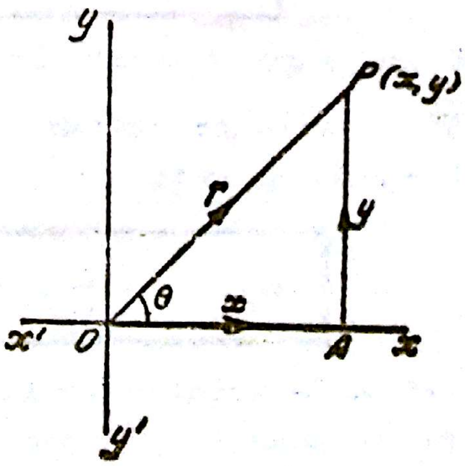

Fig. 4

Thus in Fig. 4 let xox' and yoy' be perpendicular axes where xox' — x-axis (axis of reals) yoy'—y-axis (axis of imaginaries).

Then to represent the complex number $x + iy$ measure a distance x ($\overrightarrow{OA}$) along the axis of reals and to this add a vector 'y' ($\overrightarrow{AP}$) perpendicular to the axis of reals. The resultant of these two vectors $\overrightarrow{OA}$ and $\overrightarrow{AP}$ will be a vector $\overrightarrow{OP}$ which represents a complex number $x + iy$. *Hence vector joining origin to the point* P (x, y) *represents the complex number* $x + iy$.

2·4. Polar form of complex number :—

With the vectorial representation as above, we can express the complex number in terms of *length* and *direction* of vector which represents it.

Let in Fig. 4, P (x, y) represent a complex number $x + iy$ and OP $= r$, $P\hat{O}x = \theta$.

Then $x = r\cos\theta$ and $y = r\sin\theta$.

Hence *polar equivalent* of the complex number $x + iy$ is given by

$$x + iy = r(\cos\theta + i\sin\theta)$$

$$\text{where } r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}$$

In the polar form of $x + iy$, the quantity r is known as the *modulus* of the complex number $x + iy$ and is represented as

$$\boxed{r = |x + iy| = \sqrt{x^2+y^2}} \quad \ldots \quad \ldots \quad (2)$$

and the angle θ which defines the position of the vector $(\overrightarrow{OP})$, is called the *amplitude or argument* of the complex number $x + iy$ and is given by

$$\boxed{\text{amp. of } x + iy = \theta = \tan^{-1}\frac{y}{x}} \quad \ldots \quad \ldots \quad (3)$$

If OP be turned in anticlockwise direction through multiples of 2π, the point P (x, y) for all such rotations, will have the same position as before and hence for all such positions of OP, the point P always represents the same complex number $x + iy$.

Hence the complex number $x + iy$ has various polar forms, when the amplitude is increased by multiples of 2π and hence the *general polar form* of $x + iy$ is given by

$$\boxed{r[\cos(2\pi m + \theta) + i\sin(2\pi m + \theta)]} \quad \ldots \quad \ldots \quad (4)$$

Where $2\pi m+\theta$ is known as the *general amplitude* of $x+iy$ and θ which lies between $-\pi$ and π is known as *principal* value of the amplitude.

2.5. Polar form of $x+iy$ for different signs of x,y :—

(i) $x+iy$:— $(x>0, y>0)$

The point P (x, y) will lie in the first quadrant, hence

$$r=\sqrt{x^2+y^2} \text{ and } \theta=\alpha$$

$$x+iy=\sqrt{x^2+y^2}\,[\cos\alpha+i\sin\alpha]$$

where $\tan\alpha=\dfrac{y}{x}$.

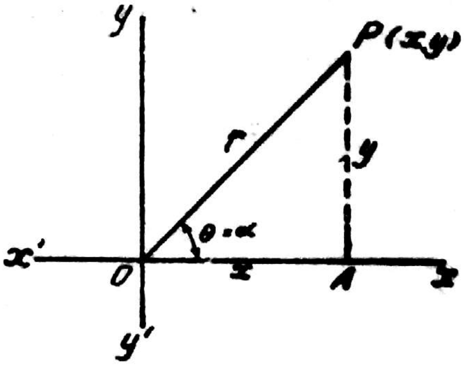

Fig. 5

Ex. $1+i\sqrt{3}=2\left[\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right]$.

(ii) $-x+iy$:—

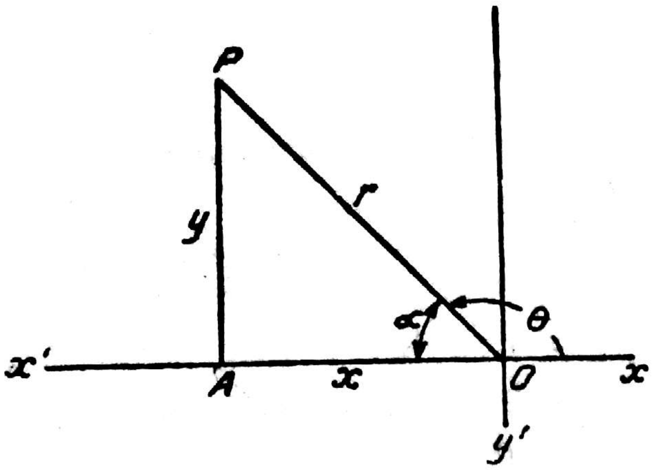

Fig. 6

The point P $(-x, y)$ lies in the second quadrant and

$A\hat{O}P=\alpha$ where $\tan\alpha=\dfrac{y}{x}$.

The amp. of complex number $=\theta=\pi-\alpha$ and $r=\sqrt{x^2+y^2}$

$$\therefore\quad x+iy=\sqrt{x^2+y^2}\,[\cos(\pi-\alpha)+i\sin(\pi-\alpha)]$$

Ex. $-1+i\sqrt{3}=2\left[\cos\left(\pi-\dfrac{\pi}{3}\right)+i\sin\left(\pi-\dfrac{\pi}{3}\right)\right]$

$$=2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right].$$

(iii) $-x-iy$:—

The point P $(-x, -y)$ lies in the third quadrant and $A\hat{O}P = \alpha$, where $\tan\alpha = \dfrac{y}{x}$ and the amp. of complex number

$$= \theta = \pi + \alpha$$

and $r = \sqrt{x^2+y^2}$

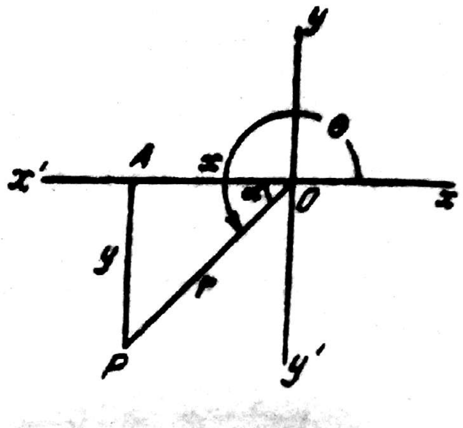

Fig. 7

$$\therefore \quad -x-iy = \sqrt{x^2+y^2}\,[\cos(\pi+\alpha) + i\sin(\pi+\alpha)]$$

Ex. $-1-i\sqrt{3} = 2\left[\cos\left(\pi+\dfrac{\pi}{3}\right) + i\sin\left(\pi+\dfrac{\pi}{3}\right)\right]$

$$= 2\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right].$$

(iv) $x-iy$:—

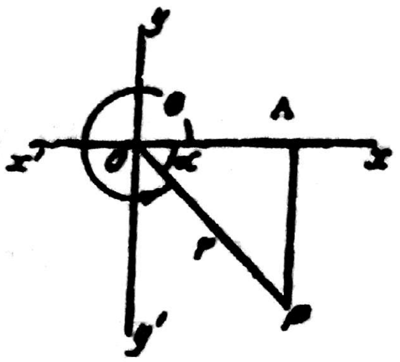

Fig. 8

The point P $(x, -y)$ lies in the fourth quadrant and $A\hat{O}P = \alpha$, where $\tan\alpha = \dfrac{y}{x}$

Amp. of complex number

$$= \theta = 2\pi - \alpha \text{ or } -\alpha$$

and $r = \sqrt{x^2+y^2}$

$$\therefore \quad x-iy = \sqrt{x^2+y^2}\,[\cos(2\pi-\alpha) + i\sin(2\pi-\alpha)].$$

$$\text{or} = \sqrt{x^2+y^2}\,[\cos\alpha - i\sin\alpha]$$

Ex. $1-i\sqrt{3} = 2\left[\cos\left(2\pi-\dfrac{\pi}{3}\right) + i\sin\left(2\pi-\dfrac{\pi}{3}\right)\right]$

$$= 2\left[\cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}\right].$$

2·6 Exponential form of complex numbers :—

When x is a real we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \text{ ad inf} \quad \dots \quad \dots \quad (5)$$

Assuming this is true for all values of x (real or complex)

Let us substitute ix for x in (5). Then,

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \dots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right)$$

$$= \cos x + i \sin x$$

as $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} \dots\dots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\dots$$

Thus the complex number $z = x + iy$ (Cartesian form)

$= r(\cos\theta + i\sin\theta)$ (Polar form)

$= re^{i\theta}$ (Exponential form)

$$\boxed{\text{Exponential form of } x + iy = re^{i\theta}} \quad \dots \quad \dots \quad (6)$$

Thus we have—

$$\text{and} \quad \boxed{\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned}} \quad \dots \quad \dots \quad (7)$$

These results are important. Although the use of the symbol e^{ix} is arbitrary, it obeys the index law. For

$$e^{ix} e^{iy} = (\cos x + i \sin x)(\cos y + i \sin y)$$

$$= \cos(x + y) + i \sin(x + y)$$

$$= e^{i(x+y)}$$

we thus take e^z, in genral, to represent the series

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \quad \dots$$

2·7 Mathematical operations with complex numbers and their representation on Argand's Diagram :—

(a) Addition and subtraction :—

Let the complex numbers be

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

then $$\boxed{z_1 \pm z_2 = (x_1 \pm x_2) + i\,(y_1 \pm y_2)} \quad \cdots \quad \cdots \quad (8)$$

Hence to add or subtract complex numbers, add or subtract the real and imaginary parts separately.

Ex. $(2-3i)+(4+6i) = (2+4)+i\,(-3+6) = 6+3i.$

Graphical represention :—

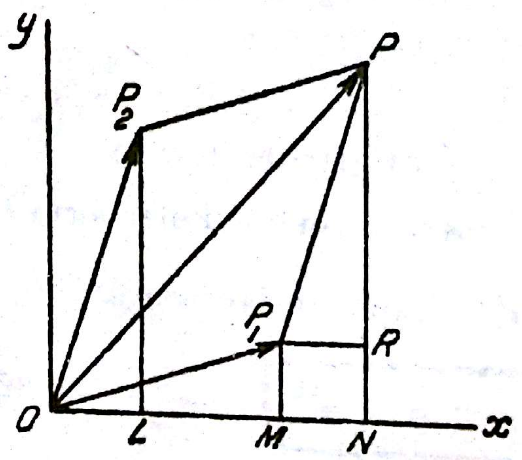

Fig. 9

Represent the complex numbers z_1, z_2 by vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. [Fig. 9]. Complete the parallelogram $OP_1\ PP_2$, then by the parallelogram law of vector addition, $\overrightarrow{OP}$ represents the sum of two complex numbers. For

$$\begin{aligned} ON &= OL + LN \\ &= OL + OM \\ &= x_1 + x_2 \end{aligned} \quad \left(\begin{array}{l} \text{Projection of } OP_1 \text{ on } x \text{ axis} \\ = \text{Projection of } P_2P \text{ on } x \text{ axis} \\ \text{as } OP_1 = \parallel \text{ to } P_2P \end{array} \right)$$

$$\text{Similarly, } \begin{aligned} PN &= RN + PR \\ &= P_1M + P_2L \\ &= y_1 + y_2 \end{aligned} \quad \left\{ \begin{array}{l} \text{Considering the projection of} \\ OP_2 \text{ and } P_1P \text{ on } y \text{ axis} \\ \text{and } P_1M = RN. \end{array} \right.$$

Thus P represents the point $(x_1 + x_2, y_1 + y_2)$, hence it represents the complex number $z_1 + z_2$.

To subtract the complex number z_2 from z_1, graphically add the vectors representing complex numbers z_1 and $-z_2$.

(b) Product of two complex numbers :—

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

then $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2$ (as $i^2 = -1$)

$$\therefore \quad \boxed{z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)} \quad \cdots \quad \cdots \quad (9)$$

This operation is more simplified by reducing z_1, z_2 to their *polar forms*. Thus

Let $x_1 + iy_1 = r_1\,(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$x_2 + iy_2 = r_2\,(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

Then

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

i. e. $\boxed{z_1 \cdot z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}}$

Hence the product of the complex numbers is a complex number whose *modulus* is the *product of their moduli* and whose *amplitude is the sum of their amplitudes.*

Thus we have

$$r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \times r_3(\cos\theta_3 + i\sin\theta_3)$$
$$= r_1 r_2 r_3 \{\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)\}$$

This can be extended, step by step, to the product of any number of factors, giving the important result for the product of any number of complex quantities viz.

modulus of the product = product of moduli of factors
amplitude of the product = sum of amplitudes of the factors.

Graphical representation (Argand's diagram) :—

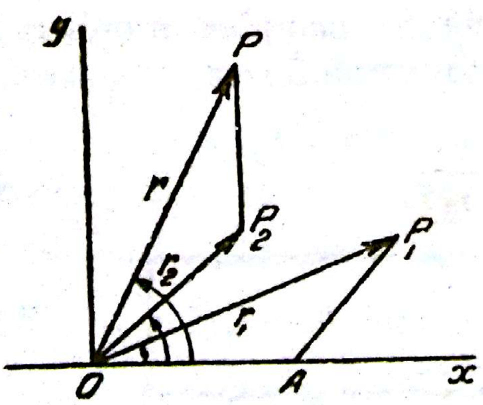

Fig. 10

Let in Fig. 10,

P_1 represent

$$z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$$

P_2 represent

$$z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)$$

and $OA = 1$.

construct the triangle OP_2P similar to the triangle OAP_1 such that

$$O\hat{P_2}P = O\hat{A}P_1 \text{ and } P\hat{O}P_2 = A\hat{O}P_1 = \theta_1$$

Then the vector $\overrightarrow{OP}$ represents the product of the complex numbers represented by P_1 and P_2.

For if P is $z = r(\cos\theta + i\sin\theta)$ we have, as ΔOP_1A and ΔOPP_2 are similar,

$$\frac{OP}{OP_1} = \frac{OP_2}{OA} \text{ i. e. } \frac{r}{r_1} = \frac{r_2}{1}$$

$$\therefore \quad r = r_1 r_2$$

$$\text{also } \theta = x\hat{O}P = x\hat{O}P_2 + P_2\hat{O}P.$$

$$= A\hat{O}P_1 + x\hat{O}P_2 = \theta_1 + \theta_2$$

Hence $z = z_1 \cdot z_2$, giving simple graphical construction for a product.

(c) Quotient of two complex numbers :—

The fact that the product of two conjugate complex numbers is a real number [viz. $(x + iy)(x - iy) = x^2 + y^2$] leads to the following method of division, where the denominator is always expressed as a real number. Thus,

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$\therefore \quad \boxed{\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i\,\frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}} \quad \ldots(11)$$

But it is more convenient to divide the complex numbers in their polar forms or better still in exponential forms.

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\therefore \quad \boxed{\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{r_1}{r_2}\left\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right\}} \quad \ldots \quad (12)$$

Hence the *modulus* of the quotient of two complex numbers is the *quotient of their moduli* and *amplitude* of the quotient is the *difference of their amplitudes*.

Graphical representation (Argand's diagram)

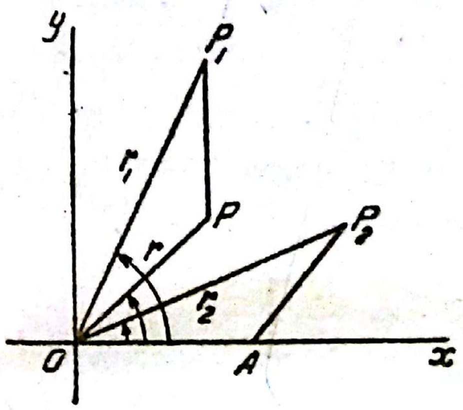

Fig. 11

Let $\theta_1 > \theta_2$

P_1 represent $r_1 (\cos \theta_1 + i \sin\theta_1)$

P_2 ,, $r_2 (\cos \theta_2 + i \sin\theta_2)$

and OA = 1 along x axis.

Construct a $\triangle$ OP_1P similar to $\triangle$ OP_2A such that

$$O\hat{P}P_1 = O\hat{A}P_2 \text{ and}$$

$$P\hat{O}P_1 = P_2O\hat{A} = \theta.$$

Then the point P thus obtained, represents the quotient of the complex numbers represented by P_1 and P_2.

For, let OP represent $r (\cos \theta + i \sin \theta)$

$\triangle$ OPP_1 is similar to $\triangle$ OAP_2

$$\therefore \frac{OP}{OA} = \frac{OP_1}{OP_2} \text{ i.e. } \frac{r}{1} = \frac{r_1}{r_2}$$

Thus $r = r_1/r_2$

$$\text{and } \theta = P\hat{O}X = P_1\hat{O}x - P_1\hat{O}P$$

$$= P_1\hat{O}x - P_2\hat{O}A \;\; (\text{by construction } P_1\hat{O}P = \hat{P_2}OA)$$

$$= \theta_1 - \theta_2$$

This gives graphical representation of $z = \dfrac{z_1}{z_2}$.

(d) Powers of complex number — De Moivre's Theorem :-

The Theorem states that *if n is any real number, one of the values of* $(\cos \theta + i \sin \theta)^n$ *is* $\cos n\theta + i \sin n\theta$.

We will consider three cases (i) n + ve integer (ii) n – ve integer and (iii) n – a fraction.

(i) Let *n be a + ve integer* :-

By actual multiplication from the result (7), we have
$(\cos \theta + i \sin \theta)^n = (\cos\theta + i \sin \theta) \times (\cos\theta + i \sin\theta) \ldots n$ times

[by law of index]

$= \cos[\theta + \theta + \ldots n \text{ times}] + i \sin[\theta + \theta + \ldots n \text{ times}]$

$= \cos n\theta + i \sin n\theta.$

(ii) Let n be a $-$ *ve integer* :–

$n = -m$, where m is a $+$ ve integer

$$\therefore\ (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} \left[\text{as } a^{-m} = \frac{1}{a^m}\right]$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \quad [\text{from case } (i)]$$

$$= \frac{(\cos m\theta - i\sin m\omega\,\theta)}{(\cos m\theta + i\sin m\theta)\,(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta.$$

(iii) Let n *be a fraction* :–

$n = \dfrac{p}{q}$, where p, q are $+$ve or $-$ve integers.

From (i) & (ii), we have

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\theta + i\sin\theta$$

$$\therefore\ (\cos\theta + i\sin\theta)^{1/q} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$$

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{p/q}$$

$$= [(\cos\theta + i\sin\theta)^{1/q}]^p$$

$$= \left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^p$$

$$= \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

$$= \cos n\theta + i\sin n\theta$$

Thus

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta} \quad \ldots \ldots \quad (13)$$

Example :- Simplify $\dfrac{(\cos 3\theta + i\sin 3\theta)^4 (\cos 5\theta - i\sin 5\theta)^{4/5}}{(\cos \frac{3}{2}\theta + i\sin \frac{3}{2}\theta)^{2/9} (\cos \frac{4}{5}\theta - i\sin \frac{4}{5}\theta)^{10}}$

Here $(\cos 3\theta + i\sin 3\theta)^4 = \{(\cos\theta + i\sin\theta)^3\}^4 = (\cos\theta + i\sin\theta)^{12}$

$(\cos 5\theta - i\sin 5\theta)^{4/5} = [\cos(-5\theta) + i\sin(-5\theta)]^{4/5}$

$= \{(\cos\theta + i\sin\theta)^{-5}\}^{4/5} = (\cos\theta + i\sin\theta)^{-4}$

$(\cos\frac{3}{2}\theta + i\sin\frac{3}{2}\theta)^{2/9} = \{(\cos\theta + i\sin\theta)^{3/2}\}^{2/9} = (\cos\theta + i\sin\theta)^{1/3}$

$(\cos\frac{4}{5}\theta - i\sin\frac{4}{5}\theta)^{10} = [\cos(-\frac{4}{5}\theta) + i\sin(-\frac{4}{5}\theta)]^{10}$

$= \{(\cos\theta + i\sin\theta)^{-4/5}\}^{10} = (\cos\theta + i\sin)^{-8}$

$\therefore$ **Exp.** $= \dfrac{(\cos\theta + i\sin\theta)^{12} (\cos\theta + i\sin\theta)^{-4}}{(\cos + i\sin\theta)^{1/3} (\cos\theta + i\sin\theta)^{-8}}$

$= (\cos\theta + i\sin\theta)^{(12 - 4 - \frac{1}{3} + 8)}$

$= (\cos\theta + i\sin\theta)^{47/3}$

$= \cos\dfrac{47}{3}\theta + i\sin\dfrac{47}{3}\theta$ by De Moivre's Theorem.

Aliter :

(Solve the Example by using $e^{i\theta} = \cos\theta + i\sin\theta$)

(e) Roots of a complex number :-

By De Moivre's theorem, we have

$$\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n = \cos\left(n.\frac{\theta}{n}\right) + i\sin\left(n.\frac{\theta}{n}\right)$$

$$= \cos\theta + i\sin\theta.$$

Hence from the meaning of a n^{th} root, we see that $\left(\cos\dfrac{\theta}{n} + i\sin\dfrac{\theta}{n}\right)$ is one of the n^{th} roots of $\cos\theta + i\sin\theta$.

Since θ may be increased by integral multiple of 2π without changing the value of $\cos\theta + i\sin\theta$, the other values of n^{th} roots are obtained as follows by expressing the number in the general polar form :

$(\cos\theta + i\sin\theta)^{1/n} = [\cos(2\pi k + \theta) + i\sin(2\pi k + \theta]^{1/n}$, where k is an integer.

$= \cos\dfrac{2\pi k + \theta}{n} + i\sin\dfrac{2\pi k + \theta}{n}$ (by De Moivre's Th.)

Hence $\cos \dfrac{2\pi k + \theta}{n} + i \sin \dfrac{2\pi k + \theta}{n}$ is the n^{th} root of $(\cos \theta + i \sin \theta)$ for any integral value of k. It can be seen that for the values of $k = 0, 1, 2, \ldots (n - 1)$, the n^{th} roots of $(\cos \theta + i \sin \theta)$ are all *different* and further values of $k = n, n + 1 \ldots$ merely give repetitions of values in the above group.

Thus there are n *different values* to the n^{th} *roots* of $(\cos \theta + i \sin \theta)$.

Hint :- *In evaluating fractional powers of a complex number express the complex number in the general polar form.*

Example 1. Find the cube roots of unity.

Since $1 = \cos (2\pi k) + i \sin (2\pi k)$

the values of $1^{1/3}$ are given by

$$(1)^{1/3} = \cos \frac{2\pi k}{3} + i \sin \frac{2\pi k}{3}$$

$\therefore$ The values are

$k = 0, (1)^{1/3} = \cos 0 + i \sin 0 = 1.$

$k = 1, (1)^{1/3} = \cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3} = -\dfrac{1}{2} + i \dfrac{\sqrt{3}}{2}$

$k = 2, (1)^{1/3} = \cos \dfrac{4\pi}{3} + i \sin \dfrac{4\pi}{3} = -\dfrac{1}{2} - i \dfrac{\sqrt{3}}{2}$

and for all other integral values of k, these values will be the repeated. The two imaginary values are denoted by ω and ω^2. Thus $\omega = \dfrac{-1 + i\sqrt{3}}{2}$,

$\omega^2 = \dfrac{-1 - i\sqrt{3}}{2}$.

Example 2. Find the two values of the sq. root of $\dfrac{1 + 2i}{2 + i}$.

$1 + 2i = \sqrt{5}\,[\cos 63° 26' + i \sin 63° 26']$

$2 + i = \sqrt{5}\,[\cos 26° 34' + i \sin 26° 34']$

$\therefore \dfrac{1 + 2i}{2 + i} = \cos 36° 52' + i \sin 36° 52'$

$= \cos [2\pi k + 36° 52'] + i \sin [2\pi k + 36° 52']$

$\therefore \sqrt{\dfrac{1 + 2i}{2 + i}} = \cos \left[\dfrac{2\pi k + 36° 52'}{2}\right] + i \sin \left[\dfrac{2\pi k + 36° 52'}{2}\right]$

When $k = 0$, the value is

$\cos 18° 26' + i \sin 18° 26' = 0{\cdot}9487 + 0{\cdot}3162\, i$

When $k = 1$, the second sq. root is given by

$\cos[180° + 18°52'] + i \sin[180° + 18°26']$

$= -0.9487 - 0.3162\,i.$

Thus the required sq. roots are $\pm(0{\cdot}9487 + 0{\cdot}3162\,i)$

Solved Problems :–

Example 1. The centre of a regular hexagon is at the origin and one vertex is given by $1 + i$ on Argand Diagram. Find the remaining vertices.

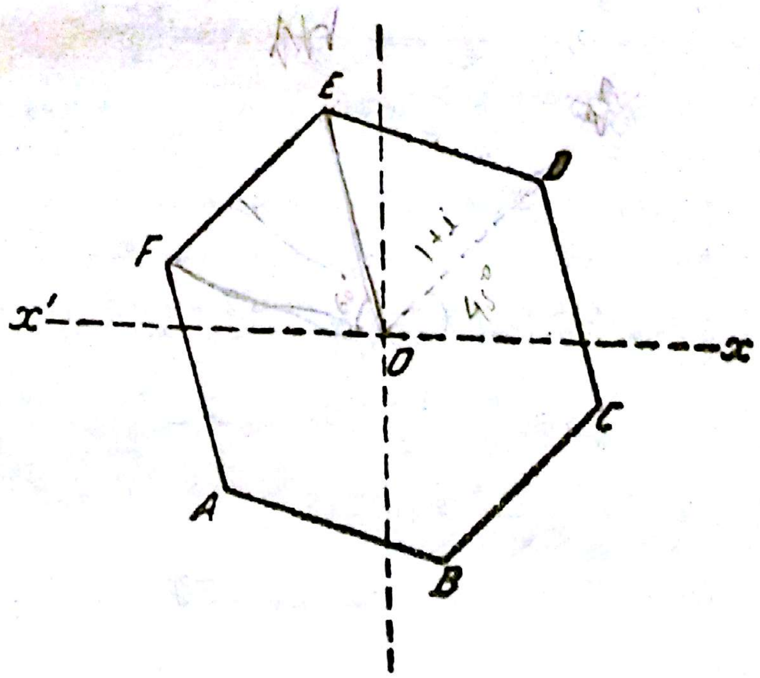

Fig. 12

As $\overrightarrow{OD}$ represents $1 + i$, OD makes an angle $\frac{\pi}{4}$ with x axis and is of length $\sqrt{2}$. Being a regular hexagon OE, OF, OA, OB and OC make angles $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ with OD. Hence the amplitudes of complex numbers representing E, F, A, B and C are

$\frac{\pi}{3} + \frac{\pi}{4}$; $\frac{2\pi}{3} + \frac{\pi}{4}$; $\pi + \frac{\pi}{4}$; $\frac{4\pi}{3} + \frac{\pi}{4}$; $\frac{5\pi}{3} + \frac{\pi}{4}$ respectively and

its moduli are $\sqrt{2}$ [as OD = OE = OF = OA = OB = OC].

$\therefore$ Vectors representiug the vertices are

For E, $\sqrt{2}\,[\cos 105° + i \sin 105°] = -(0.3660) + (1{\cdot}366)\,i$

For F, $\sqrt{2}\,[\cos 165° + i \sin 165°] = -(1.366) + (0.366)\,i$

For A, $\sqrt{2}\,[\cos 225° + i \sin 225°] = -1 - i$

For B, $\sqrt{2}\,[\cos 285° + i \sin 285°] = (0.366) - (1{\cdot}366)\,i$

For C, $\sqrt{2}\ [\cos 345^\circ + i \sin 345^\circ] = (1{\cdot}366) - (0.366)\, i$

Example 2. If $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \gamma + i \sin \gamma$ and $u = \cos \delta + i \sin \delta$, prove

(i) $$xy + zu = 2 \cos \frac{\alpha + \beta - \gamma - \delta}{2} \left[\cos \frac{\alpha + \beta + \gamma + \delta}{2} + i \sin \frac{\alpha + \beta + \gamma + \delta}{2} \right].$$

(ii) $$\frac{1}{(x-y)(z-u)} = -\frac{1}{4} \operatorname{cosec} \frac{\alpha - \beta}{2} \operatorname{cosec} \frac{\gamma - \delta}{2} \left[\cos \frac{\alpha+\beta+\gamma+\delta}{2} - i \sin \frac{\alpha+\beta+\gamma+\delta}{2} \right]$$

(i) $$xy+zu = [\cos(\alpha+\beta) + i \sin(\alpha+\beta)] + [\cos(\gamma+\delta) + i \sin(\gamma+\delta)]$$

$$= \{\cos(\alpha+\beta) + \cos(\gamma+\delta)\} + i\{\sin(\alpha+\beta) + \sin(\gamma+\delta)\}$$

$$= \left\{ 2 \cos \frac{\alpha + \beta + \gamma + \delta}{2} \cos \frac{\alpha + \beta - \gamma - \delta}{2} \right\} + i \left\{ 2 \sin \frac{\alpha + \beta + \gamma + \delta}{2} \cos \frac{\alpha + \beta - \gamma - \delta}{2} \right\}$$

$$= 2 \cos \frac{\alpha+\beta-\gamma-\delta}{2} \left\{ \cos \frac{\alpha + \beta+\gamma+\delta}{2} + i \sin \frac{\alpha + \beta + \gamma + \delta}{2} \right\}$$

(ii) $$x - y = (\cos \alpha - \cos \beta) + i(\sin \alpha - \sin \beta)$$

$$= \left(-2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \right) + i \left(2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} \right)$$

$$= 2 i \sin \frac{\alpha - \beta}{2} \left[\cos \frac{\alpha + \beta}{2} - \frac{1}{i} \sin \frac{\alpha + \beta}{2} \right]$$

$$= 2 i \sin \frac{\alpha-\beta}{2} \left[\cos \frac{\alpha + \beta}{2} + i \sin \frac{\alpha + \beta}{2} \right]$$

Similarly,

$$(z-u) = 2 i \sin \frac{\gamma - \delta}{2} \left[\cos \frac{\gamma + \delta}{2} + i \sin \frac{\gamma + \delta}{2} \right]$$

$$\therefore \frac{1}{(x-y)(z-u)} = \frac{1}{-4 \sin \frac{\alpha - \beta}{2} \sin \frac{\gamma - \delta}{2}} \times \frac{1}{\left[\cos \frac{\alpha + \beta + \gamma + \delta}{2} + i \sin \frac{\alpha + \beta + \gamma + \delta}{2} \right]}$$

$$= -\frac{1}{4}\operatorname{cosec}\frac{\alpha-\beta}{2}\operatorname{cosec}\frac{\gamma-\delta}{2}\left[\cos\frac{\alpha+\beta+\gamma+\delta}{2} + i\sin\frac{\alpha+\beta+\gamma+\delta}{2}\right]^{-1}$$

$$= -\frac{1}{4}\operatorname{cosec}\frac{\alpha-\beta}{2}\operatorname{cosec}\frac{\gamma-\delta}{2}\left[\cos\frac{\alpha+\beta+\gamma+\delta}{2} - i\sin\frac{\alpha+\beta+\gamma+\delta}{2}\right].$$

by De Moivre's Theorem

Example 3. Simplify

$$\{(\cos\theta-\cos\phi)+i(\sin\theta-\sin\phi)\}^n+\{(\cos\theta-\cos\phi)-i(\sin\theta-\sin\phi)\}^n$$

Let $A = \cos\theta - \cos\phi = -2\sin\frac{\theta-\phi}{2}\sin\frac{\theta+\phi}{2}$

$B = \sin\theta - \sin\phi = 2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}$

$$\therefore A + iB = 2\sin\frac{\theta-\phi}{2}\left[-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right]$$

$$= 2\sin\frac{\theta-\phi}{2}\left[\cos\frac{\pi+\theta+\phi}{2} + i\sin\frac{\pi+\theta+\phi}{2}\right]$$

[In order to apply De Moivre's theorem it is essential to express a complex number in the form $\cos\theta + i\sin\theta$ and hence the conversion is effected in the second step above.]

$$\therefore (A+iB)^n = 2^n \cdot \sin^n\frac{\theta-\phi}{2}\left[\cos n\left(\frac{\pi+\theta+\phi}{2}\right) + i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)\right]$$

by De Moivre's theorem.

and $$(A-iB)^n = \left\{2\sin\frac{\theta-\phi}{2}\left[-\sin\frac{\theta+\phi}{2} - i\cos\frac{\theta+\phi}{2}\right]\right\}^n$$

$$= \left\{2\sin\frac{\theta+\phi}{2}\left[\cos\left(\frac{\pi+\theta+\phi}{2}\right) - i\sin\left(\frac{\pi+\theta+\phi}{2}\right)\right]\right\}^n$$

$$= 2^n\sin^n\frac{\theta-\phi}{2}\left[\cos n\left(\frac{\pi+\theta+\phi}{2}\right) - i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)\right]$$

by De Moivre's theorem.

$\therefore$ Expression $= (A + iB)^n + (A - iB)^n$

$$= 2^n \sin^n \frac{\theta - \phi}{2} \left[2 \cos n \left(\frac{\pi + \theta + \phi}{2} \right) \right]$$

$$= 2^{n+1} \sin^n \frac{\theta - \phi}{2} \cos n \left(\frac{\pi + \theta + \phi}{2} \right)$$

Example 4. If $z = -1 + i\sqrt{3}$ and n is an integer, prove that $2^{2n} + 2^n z^n + z^{2n}$ is zero if n is not a multiple of 3.

$$z = -1 + i\sqrt{3} = 2\left[\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right]$$

$$\therefore \quad z^n = 2^n \left[\cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3} \right]$$

Let $u = z^n = 2^n \left[\cos \dfrac{2n\pi}{3} + i \sin \dfrac{2n\pi}{3} \right]$

$v = 2^n$

Hence

$$z^{2n} + 2^n z^n + 2^{2n} = u^2 + uv + v^2$$

$$= \frac{u^3 - v^3}{u - v}$$

$$= \frac{2^{3n} [\cos 2n\pi + i \sin 2n\pi] - 2^{3n}}{2^n \left[\cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3} \right] - 2^n}$$

$$= 2^{2n} \frac{i \sin 2n\pi}{\left(\cos \frac{2n\pi}{3} - 1 \right) + i \sin \frac{2n\pi}{3}}$$

$= 0$ if n is not multiple of 3.

If n is multiple of 3, the expression takes the form $\dfrac{0}{0}$, hence is indeterminate.

Example 5. If $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$, prove that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$$

and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$

We know that if $a + b + c = 0$

then $a^3 + b^3 + c^3 = 3abc$ .. .. (i)

Let $a = \cos\alpha + i \sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$

thus we have $a + b + c = 0$

From (i) above, we have

$$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

So that by De Moivre's Theorem

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= 3\cos(\alpha + \beta + \gamma) + 3i\sin(\alpha + \beta + \gamma).$

Hence by equating real and imaginary parts, we have the required results.

Example 6. Use De Moivre's Theorem to prove the following :–

(i) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii) $\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

(i) $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + \frac{4.3}{1.2}\cos^2\theta(i\sin\theta)^2$$

$$+ \frac{4.3.2}{1.2.3}\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta)$$

$$+ i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

as $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.

So that by equating the real parts, we get

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii) $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

$$= \cos^6\theta + 6\cos^5\theta(i\sin\theta) + \frac{6.5}{1.2}\cos^4\theta(i\sin\theta)^2$$

$$+ \frac{6.5.4}{1.2.3}\cos^3\theta(i\sin\theta)^3 + \frac{6.5.4.3}{1.2.3.4}\cos^2\theta(i\sin\theta)^4$$

$$+ \frac{6.5.4.3.2}{1.2.3.4.5}\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$$

[as $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$]

$$= [\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta]$$

$$+ i[6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta]$$

By equating the imaginary parts, we get the required result.

Example 7. Expand

(i) $\cos^8\theta$ in a series of cosines of multiples of θ

(ii) $\sin^5\theta$ in a series of sines of multiples of θ

(i) Let $x + \dfrac{1}{x} = 2\cos\theta$

$$\therefore (2\cos\theta)^8 = \left(x + \frac{1}{x}\right)^8$$

$$= x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56\frac{1}{x^2} + 28\frac{1}{x^4} + 8\frac{1}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 \quad \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70.$$

[If $x = \cos\theta + i\sin\theta$, $x^m + \frac{1}{x^m} = 2\cos m\theta$]

$$= 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70$$

$\therefore\ 2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$

(ii) If $x = \cos\theta + i\sin\theta$, $x^{-1} = \cos\theta - i\sin\theta$

$$\therefore\ x - \frac{1}{x} = 2i\sin\theta$$

Hence

$$2^5 i^5 \sin^5\theta = \left(x - \frac{1}{x}\right)^5$$

$$= x^5 - 5x^3 + 10x - 10\frac{1}{x} + 5\frac{1}{x^3} - \frac{1}{x^5}$$

$$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$\left[\text{ if } x = \cos\theta + i\sin\theta,\ x^m - \frac{1}{x^m} = 2i\sin m\theta\right]$

$$= 2i\sin 5\theta - 5(2i)\sin 3\theta + 10(2i)\sin\theta.$$

$\therefore\ 2^4 i^4 \sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$$\therefore\ \sin^5\theta = \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$$

Example 8. Find the continued product of the four values of

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$$

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$= \left[\cos\left(2\pi k + \frac{\pi}{3}\right) + \sin\left(2\pi k + \frac{\pi}{3}\right)\right]$$

(general polar form is required as fractional powers involved)

$$\therefore\ \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4} = \cos\frac{\cancel{3}}{4}\left(\frac{6\pi k + \pi}{\cancel{3}}\right) + i\sin\frac{3}{4}\left(\frac{6\pi k + \pi}{3}\right)$$

$\therefore$ Different values given by $k = 0, 1, 2, 3$ are as follows :—

$$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\ \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4},\ \cos\frac{13\pi}{4} + i\sin\frac{13\pi}{4}$$

and $\cos\dfrac{19\pi}{4} + i\sin\dfrac{19\pi}{4}$

$\therefore$ Continued product of the four values (by product theorem)

$$= \cos\left(\frac{\pi}{4} + \frac{7\pi}{4} + \frac{13\pi}{4} + \frac{19\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{7\pi}{4} + \frac{13\pi}{4} + \frac{19\pi}{4}\right)$$

$$= \cos 10\pi + i\sin 10\pi = 1$$

Example 9. Solve the equation $x^9 - x^5 + x^4 - 1 = 0$

The equation is $(x^5 + 1)(x^4 - 1) = 0$

Taking the first factor, we have

$$x^5 = -1 = \cos(2k + 1)\pi + i\sin(2k + 1)\pi$$

$$\therefore\ x = [\cos(2\pi k + \pi) + i\sin(2\pi k + \pi)]^{1/5}$$

$$\therefore\ x = \cos\frac{(2k+1)\pi}{5} + i\sin\frac{(2k+1)\pi}{5}$$

giving k the values 0, 1, 2, 3, 4 successively we get the solutions $\cos 36° + i\sin 36°$, $\cos 108° + i\sin 108°$, $\cos 180° + i\sin 180°$ (i. e -1) $\cos 252° + i\sin 252°$ and $\cos 324° + i\sin 324°$

Taking the second factor, we get

$$x^4 = 1 = \cos 2\pi k + i\sin 2\pi k$$

$$\therefore\ x = \cos\frac{\pi k}{2} + i\sin\frac{\pi k}{2}$$

Giving k the values 0 1, 2 and 3 we get the solutions as $1, i, -1$ and $-i$.

Thus all roots are known.

Examples : II—A

(1) Simplify in the form $(a + ib)$:—

(i) $\dfrac{(1+i)^6 (1 - i\sqrt{3})^4}{(1-i)^8 (1 + i\sqrt{3})^5}$ (ii) $3\left(\dfrac{1+i}{1-i}\right)^2 - 2\left(\dfrac{1-i}{1+i}\right)^3$

[Ans (i) $\dfrac{i}{4}$ (ii) $-3-2i$]

(2) If $z_1 = 1 + 2i$, $z_2 = 2 + i$, show on Argand's diagram the points representing $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$ and $\dfrac{z_1}{z_2}$. Measure on the diagram the amplitude and modulus of each of these four quantities and check with the results obtained from calculations.

(3) If the corresponding vertices of two similar triangles are represented by complex numbers a_1, a_2, a_3 and b_1, b_2, b_3, then show that

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

(4) Complex numbers z_1, z_2, z_3 are represented on Argand's diagram by the points A, B, C in counterclockwise order, prove that the necessary and sufficient condition that the triangle A B C may be equilateral triangle is

$$z_1 - z_2 = e^{i\pi/3}\,(z_3 - z_2).$$

(5) If p is real and the complex number $\dfrac{1+i}{2+ip} + \dfrac{2+3i}{3+i}$ is represented in the Argand's diagram by a point on the line $y=x$, show that

$$p = -5 \pm \sqrt{21}.$$

(6) If $\arg(z+1) = \dfrac{\pi}{6}$ and $\arg.(z-1) = \frac{2}{3}\pi$, find z.

$$\left[\text{Ans. } \tfrac{1}{2} + i\,\frac{\sqrt{3}}{2}\right]$$

(7) Find z so that

$$|z+i| = |z| \text{ and } \arg.\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$$

$$\left[\text{Ans. } z = \frac{1}{2}\left(-i + \cot\frac{\pi}{8}\right)\right]$$

(8) If $\dfrac{z+i}{z+2}$ is purely imaginary, show that the locus of (x, y) is a circle of radius $\sqrt{5}/2$.

(9) If $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = \sqrt{3} - 2i$, evaluate

(i) $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$, (ii) $R_e\,(2z_1^2 + 3z_2^2 - z_3^2)$, (iii) $Im\left(\dfrac{z_1 z_2}{z_3}\right)$

$$\left[\text{Ans. (i) } \frac{3}{5} \quad \text{(ii) } -35 \quad \text{(iii) } \frac{6\sqrt{3}+4}{7}\right]$$

(10) If $x^2 + y^2 = 1$, show that

$$\frac{1+x+iy}{1+x-iy} = x + iy;\quad \frac{1+y+ix}{1+y-ix} = y + ix.$$

(11) If $z_1\ z_2$ are the roots of the equation $az^2 + bz + c = 0$, where a, b, c, are real numbers with $b^2 - 4ac < 0$. obtain the value of the expression $z_1^n + z_2^n$ in terms of a, b, c, if n is any integer.

(12) If $x = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$, prove that $(x^3 - x^3)(x^4 - x) = \sqrt{5}$.

(13) Express $\frac{1}{\sqrt{2}}(1+i)$ in the form $r(\cos\theta + i\sin\theta)$, and hence simplify :—

(i) $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{4/3}$

(ii) $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{10} + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^{10}$

[Ans. (i) $\frac{1}{2}(1+\sqrt{3}\,i)$ (ii) zero].

(14) Prove that

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n \text{ has the value } -1$$

if $n = 3k \pm 1$; and 2 if $n = 3k$ where k is an integer.

(15) If $(x_1 + iy_1)(x_2 + iy_2)\ldots(x_n + iy_n) = X + iY$, then prove that

$$\tan^{-1}\frac{y_1}{x_1} + \tan^{-1}\frac{y_2}{x_2} + \ldots + \tan^{-1}\frac{y_n}{x_n} = \tan^{-1}\frac{Y}{X}$$

(16) If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$

prove that one value of $x^p y^q + \frac{1}{x^p y^q}$ is

$$2\cos(p\theta + q\phi).$$

(17) Prove that $\sqrt[n]{x+iy} + \sqrt[n]{x-iy}$ has n real values and find those of

$$\sqrt[3]{1+i\sqrt{3}} + \sqrt[3]{1-i\sqrt{3}}$$

$$\left[\text{Ans. } 2^{4/3}\cos\frac{(6k+1)\pi}{3}\right]$$

(18) Find all the values of—

(i) $(36+64i)^{1/2}$ (ii) $\left(\frac{2+3i}{1+i}\right)^{1/2}$ (iii) $\left(\frac{1}{1+i}\right)^{1/3}$ (iv) $i^{1/3}$

(v) $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{1/4}$ (vi) $(\sqrt{3} - i)^{2/5}$ (vii) $16^{1/4}$

[Ans. (i) $\pm(7.397 + 4.326i)$ (ii) $\pm(1.5889 + 0.1571i)$

(iii) $0.8606 - 0.2306i$; $-0.6300 - 0.6300i$;

$-0.2306 + 0.8606i$

(iv) $-i, \dfrac{\pm\sqrt{3}+i}{2}$ (v) $\pm\dfrac{i+\sqrt{3}}{2}$; $\pm\dfrac{i\sqrt{3}-1}{2}$

(vi) $\sqrt[5]{4}\left[\cos\dfrac{k\pi}{15}+i\sin\dfrac{k\pi}{15}\right]$, where,

[$k = -1, 5, 11, 17$ or 23 (vii) ± 2 and $\pm 2i$]

(19) Find the three cube roots of $1-\cos\theta - i\sin\theta$ when $0 < \theta < 2\pi$,

$$\left[\text{Ans. } r = \left(2\sin\frac{\theta}{2}\right)^{1/3}, \phi = \frac{\theta}{6} \pm \frac{\pi}{3}, \phi = \frac{\theta}{6} - \pi\right]$$

where root $= r(\cos\phi + i\sin\phi)$

(20) Prove that

$$(x+iy)^{\frac{m}{n}} + (x-iy)^{\frac{m}{n}} = 2(x^2+y^2)^{\frac{m}{2n}} \cos\left(\frac{m}{n}\tan^{-1}\frac{y}{x}\right)$$

(21) Solve the equations :–

(i) $x^7 + x^4 + x^3 + 1 = 0$

(ii) $x^7 + 1 = 0$

(iii) $x^9 + x^6 + x^3 + 1 = 0$

(iv) $x^{10} + 11x^5 + 10 = 0$.

Ans. :– (i) $-1, \cos\dfrac{\pi}{3} \pm i\sin\dfrac{\pi}{3}, \pm\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$

and $\pm\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

(ii) -1 and $\cos\dfrac{k\pi}{7} \pm i\sin\dfrac{k\pi}{7}$ when $k = 1, 3$ or 5

(iii) $\pm i$; $\cos\dfrac{\pi}{6} \pm i\sin\dfrac{\pi}{6}$; $\cos\dfrac{5\pi}{6} \pm i\sin\dfrac{5\pi}{6}$;

$-1, \cos\dfrac{\pi}{3} \pm i\sin\dfrac{\pi}{3}$.

(iv) $\cos\dfrac{\pi}{5} \pm i\sin\dfrac{\pi}{5}$; $\cos\dfrac{3\pi}{5} \pm i\sin\dfrac{3\pi}{5}$; -1;

$\sqrt[5]{10}\left[\cos\dfrac{\pi}{5} \pm i\sin\dfrac{\pi}{5}\right]$;

$\sqrt[5]{10}\left[\cos\dfrac{3\pi}{5} \pm i\sin\dfrac{3\pi}{5}\right]$ and $-\sqrt[5]{10}$

(22) Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation

$$x^4 + x^2 + 1 = 0$$

$\left[\text{Ans. } \pm 1, \pm i, \pm\left(\cos\dfrac{\pi}{6} \pm i\sin\dfrac{\pi}{6}\right)\right.$ and

$$\pm \left(\cos \frac{\pi}{3} \pm i \sin \frac{\pi}{3} \right). \text{ The last four values} \Big]$$

(23) Solve the equation $z^2 + z^{-2} = i$

$$\left[\pm \frac{1}{2} (1+i)(1+\sqrt{5})^{1/2}, \pm \frac{1}{2} (1-i)(\sqrt{5}-1)^{1/2} \right]$$

(24) Express $\dfrac{1}{(x+iy)^2} + \dfrac{1}{(x-iy)^2}$ in the form $a+ib$ giving the values of a, b in terms of x and y.

$$\left[\text{Ans. } a = \frac{2(x^2-y^2)}{(x^2+y^2)^2};\ b = 0 \right]$$

(25) If n is a positive integer, show that

$$(1+i)^n + (1-i)^n = (\sqrt{2})^{n+2} \cos\left(\frac{n\pi}{4}\right)$$

and show that the continued product of all the values of $(1+i)^{1/5}$ is $1+i$.

(26) Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$.

(27) Prove tha:

$$(1+\cos\theta + i\sin\theta)^n = 2^n \cos^n \frac{\theta}{2} \left(\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2} \right)$$

(28) Use Demoivre's Theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$

Deduce that $5 \tan^4 \dfrac{\pi}{10} - 10 \tan^2 \dfrac{\pi}{10} + 1 = 0$

2.8 Circular functions of complex angles :—

From the formulae (13) in the preceding paragraph for e^{ix} and e^{-ix}, we have

$$e^{ix} + e^{-ix} = 2\cos x, \quad e^{ix} - e^{-ix} = 2i\sin x$$

whence

$$\boxed{\begin{aligned} \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\ \cos x &= \frac{e^{ix} + e^{-ix}}{2} \end{aligned}} \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

These are known as exponential values of the sine and cosine.

For any non-real quantity z, where the geometrical definitions of $\sin z$, $\cos z$ no longer have a meaning, we may regard them as defined as above so that,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \tan z = \frac{\sin z}{\cos z}$$

with cosec z, sec z, cot z as their respective reciprocals.

2·9 Definition of hyperbolic functions :–

From the analogy to the definitions of sin x and cos x, we define the new functions known as *hyperbolic functions.*

$$\boxed{\begin{aligned} \text{Hyperbolic sin of } x &= \sinh x = \frac{e^x - e^{-x}}{2} \\ \text{Hyperbolic cosine of } x &= \cosh x = \frac{e^x + e^{-x}}{2} \end{aligned}} \quad \ldots\ldots (15)$$

$$\text{and } \tanh x = \frac{\sinh x}{\cosh x}$$

and cosech x, sech x, coth x are defined as the reciprocals of sinh x, cosh x, and tanh x respectively.

2.10 Relations between circular and hyperbolic functions :–

Using the definitions of sin z and cos z, we have

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i}$$

$$= \frac{e^{-x} - e^{x}}{2i} = \frac{-1}{i}\left[\frac{e^x - e^{-x}}{2}\right]$$

$$= i \sinh x$$

$$\text{and } \cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2}$$

$$= \frac{e^{-x} + e^{x}}{2} = \cosh x$$

Hence, we have

$$\boxed{\begin{aligned} \sin(ix) &= i \sinh x \\ \cos(ix) &= \cosh x \\ \tan(ix) &= i \tanh x \end{aligned}} \quad \ldots \quad \ldots (16)$$

These definitions enable us to deduce the properties of hyperbolic functions from those of circular functions. It may be verified that the properties of sin z, cos z, tan z etc. when z is not real may be deduced from definitions given above.

(1) $\cos^2 z + \sin^2 z = \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2$

$$= \frac{e^{2iz} + 2 + e^{-2iz}}{4} - \frac{e^{2iz} - 2 + e^{-2iz}}{4}$$

$$= \frac{4}{4} = 1 \ (\text{as } i^2 = -1)$$

(2) In the above write $z = ix$, then

$\cos^2 (ix) + \sin^2 (ix) = 1,$

i. e. $\cosh^2 x + (i \sinh x)^2 = 1$

hence $\cosh^2 x - \sinh^2 x = 1$

(3) $\sin (z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

put $z_1 = ix$ and $z_2 = iy$

$\therefore \sin i (x \pm y) = \sin (ix) . \cos (iy) \pm \cos (ix) \sin (iy)$

i. e. $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

Similarly from the expansion of $\cos (z_1 \pm z_2)$, we get

$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

We have following formulae for hyperbolic function which can be deduced from those of circular functions by similar methods as illustrated above.

(i) $\tanh (x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(ii) $\sinh x + \sinh y -- 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

$\sinh x - \sinh y = 2 \sinh \dfrac{x-y}{2} \cosh \dfrac{x+y}{2}$

$\cosh x + \cosh y = 2 \cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

$\cosh x - \cosh y = 2 \sin \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

(iii) $\cosh^2 x = \frac{1}{2} (1 + \cosh 2x)$

$\sinh^2 x = \frac{1}{2} (\cosh 2x - 1)$

2.11. Inverse hyperbolic functions :—

If $x = \cosh y$ then we write $y = \cosh^{-1} x$

If x be real, we have

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

so that $e^{2y} - 2xe^y + 1 = 0$

and hence $e^y = x \pm \sqrt{x^2 - 1}$

$$\therefore \quad \boxed{y = \cosh^{-1} x = \pm \log (x + \sqrt{x^2 - 1})} \quad \ldots \quad (17)$$

The positive value of the right hand side is the one always taken.

Similarly it can be shown, if x is real

$$\boxed{\begin{aligned} \sinh^{-1} x &= \log (x + \sqrt{x^2+1}) \\ \tanh^{-1} x &= \tfrac{1}{2} \log \frac{1+x}{1-x} \end{aligned}} \quad \ldots \ldots \quad (18)$$

2·12. Differentiation and Integration :—

(i) $y = \cosh x, \quad \dfrac{dy}{dx} = \sinh x, \quad \therefore \int \sinh x \, dx = \cosh x$

(ii) $y = \sinh x, \quad \dfrac{dy}{dx} = \cosh x, \quad \therefore \int \cosh x \, dx = \sinh x$

(iii) $y = \tanh x, \quad \dfrac{dy}{dx} = \operatorname{sech}^2 x, \quad \therefore \int \operatorname{sech}^2 x \, dx = \tanh x$

(iv) $y = \sinh^{-1} \dfrac{x}{a}, \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{a^2 + x^2}}$

$$\therefore \int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a}.$$

(v) $y = \cosh^{-1} \dfrac{x}{a}, \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - a^2}}$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a}$$

(vi) $y = \tanh^{-1} \dfrac{x}{a}, \quad \dfrac{dy}{dx} = \dfrac{a}{a^2 - x^2}$

$$\therefore \int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

(vii) $y = \operatorname{cosech}^{-1} \dfrac{x}{a}, \quad \dfrac{dy}{dx} = \dfrac{-a}{x\sqrt{x^2 + a^2}}$

$$\therefore \int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a} \operatorname{cosech}^{-1} \frac{x}{a}$$

(viii) $y = \operatorname{sech}^{-1} \dfrac{x}{a}, \quad \dfrac{dy}{dx} = \dfrac{-a}{x\sqrt{a^2 - x^2}}$

$$\therefore \int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \operatorname{sech}^{-1} \frac{x}{a}$$

Series for cosh x and sinh x :—

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \quad \dots$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad \dots$$

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) = x + \frac{x^2}{3!} + \frac{x^5}{5!} + \dots \quad \dots$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots \dots}{1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots \dots \dots}$$

$$= x - \tfrac{1}{3}x^3 + \frac{2}{15}x^5 \quad \dots \dots \dots$$

Thus :—

$$\boxed{\begin{aligned} \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\ \tanh x &= x - \tfrac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \end{aligned}} \quad \dots \dots \quad (19)$$

2·13. Graphs of the Hyperbolic Functions :—

The values of sinh x, cosh x, tanh x for $x = -\infty$ 0, $+\infty$ are obtained from the definitions and are tabulated thus :—

x	sinh x	cosh x	tanh x
$-\infty$	$-\infty$	$+\infty$	-1
0	0	1	0
$+\infty$	$+\infty$	$+\infty$	1

Since e^x, e^{-x} are always positive (or zero), it will be seen that tanh x is always < 1, its value must lie between ± 1.

From the definitions it is obvious that

$$\sinh x < \cosh x$$

Hence the curves $y = \sinh x$, $y = \cosh x$, $y = \tanh x$ have the forms shown in the Fig. 13.

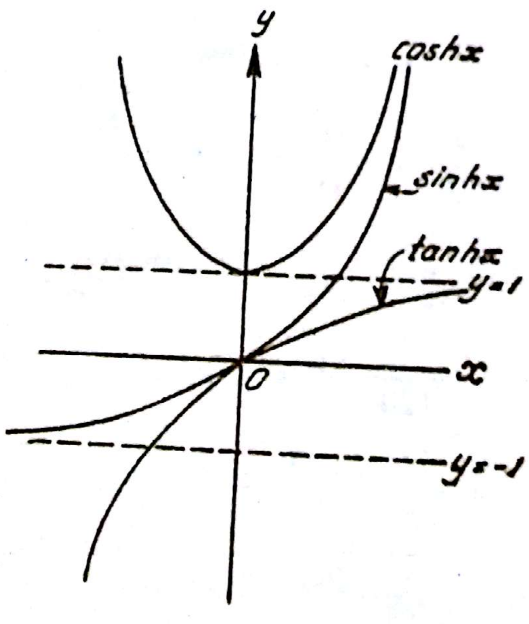

Fig. 13

These curves can be plotted more accurately from the tables of hyperbolic functions.

The curves $y = \text{cosech} x$, $y = \text{sech } x$, $y = \coth x$ can be sketched as reciprocal of those shown above, but they are not used to any great extent.

Equations involving hyperbolic functions can be solved approximately by graphical methods.

2.14. Separation of real and imaginary parts of the circular and hyperbolic functions of complex variable :-

(a) $\sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$

$$= \sin x \cosh y + i \cos x \sinh y$$

Similarly, $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.

(b) $\tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)}$

$$= \frac{2\sin(x+iy)}{2\cos(x+iy)} \cdot \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos(2iy)}$$

$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\,\frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Similarly, $\tan(x - iy)$

$$= \frac{\sin 2x}{\cos 2x + \cosh 2y} - i\,\frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Thus if $\tan(x+iy) = p + iq$

then $\tan(x-iy) = p - iq$

(c) $\sinh(x+iy) = \sinh x \cosh(iy) + \cosh x \sinh iy$

$\left[\begin{array}{l} i\sinh iy = \sin i(iy) = -\sin y \quad \therefore \sinh iy = i\sin y \\ \text{and } \cosh iy = \cos i(iy) = \cos y \end{array}\right]$

$$= \sinh x \cos y + i \cosh x \sin y$$

similarly, $\cosh(x+iy)$ can be expressed in the form $a+ib$

(d) $\tanh(x+iy)$:—

$\tanh(x+iy) = \dfrac{\sinh(x+iy)}{\cosh(x+iy)}$

$$= \frac{2\sinh(x+iy)\cosh(x-iy)}{2\cosh(x+iy)\cosh(x-iy)}$$

$$= \frac{\sinh 2x + \sinh(2iy)}{\cosh 2x + \cosh(2iy)}$$

$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y} + i\,\frac{\sin 2y}{\cosh 2x + \cos 2y}$$

Similarly,

$$\tanh(x - iy) = \frac{\sinh 2x}{\cosh 2x + \cos 2y} - i\,\frac{\sin 2y}{\cosh 2x + \cos 2y}$$

In all the problems involving separation of real and taginary parts of tan ($\alpha + i\beta$) proceed as follows :-

Let $\tan(\alpha + i\beta) = x + iy$

then $\tan(\alpha - i\beta) = x - iy$ (as proved)

Then to express α, β in terms of x and y, we have

$$\tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$$

$$= \frac{\tan(\alpha + i\beta) + \tan(\alpha - i\beta)}{1 - \tan(\alpha + i\beta)\tan(\alpha - i\beta)}$$

$$= \frac{(x + iy) + (x - iy)}{1 - x^2 - y^2} = \frac{2x}{1 - x^2 - y^2}$$

which expresses α in terms of x and y.

Similarly,

$$\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$$

$$= \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta).\tan(\alpha - i\beta)}$$

$$= \frac{(x + iy) - (x - iy)}{1 + x^2 + y^2}$$

$$\therefore\ i\tanh 2\beta = \frac{2iy}{1 + x^2 + y^2}$$

$$\therefore\ \tanh 2\beta = \frac{2y}{1 + x^2 + y^2}$$

which give β in terms of x and y.

Thus in problems when

$\tan(\alpha + i\beta) = x + iy$ is given, then use

$\tan(\alpha - i\beta) = x - iy$

and with combinations as illustrated above, we get α, β in terms of x and y. This method will be clear from the problems solved here.

Example 1. Separate into its real and imaginary parts the expression $\tan^{-1}(\alpha + i\beta)$.

Let $\tan^{-1}(\alpha + i\beta) = x + iy$

$\therefore \quad \tan(x + iy) = \alpha + i\beta$

and $\tan(x - y) = \alpha - \beta$

$$\therefore \quad \tan 2x = \tan\{(x + iy) + (x - iy)\}$$

$$= \frac{(\alpha + i\beta) + (\alpha - i\beta)}{1 - (\alpha + i\beta)(\alpha - i\beta)}$$

$$= \frac{2\alpha}{1 - \alpha^2 - \beta^2}$$

$$\therefore \quad x = \tfrac{1}{2} \tan^{-1} \frac{2\alpha}{1 - \alpha^2 - \beta^2}$$

$$\text{and } \tan(2iy) = \tan[(x + iy) - (x - iy)]$$

$$= \frac{(\alpha + i\beta) - (\alpha - i\beta)}{1 + (\alpha + i\beta)(\alpha - i\beta)}$$

$$\therefore \quad i \tanh 2y = \frac{2i\beta}{1 + \alpha^2 + \beta^2}$$

$$\therefore \quad \tanh 2y = \frac{2\beta}{1 + \alpha^2 + \beta^2}.$$

$$\therefore \quad 2y = \tanh^{-1} \frac{2\beta}{1 + \alpha^2 + \beta^2} = \tfrac{1}{2} \log \left\{ \frac{1 + \dfrac{2\beta}{1 + \alpha^2 + \beta^2}}{1 - \dfrac{2\beta}{1 + \alpha^2 + \beta^2}} \right\}$$

$$\left[\text{as } \tanh^{-1} x = \tfrac{1}{2} \log \frac{1 + x}{1 - x} \right]$$

$$\therefore \quad y = \tfrac{1}{4} \log \frac{(1 + \beta)^2 + \alpha^2}{(1 - \beta)^2 + \alpha^2}$$

$$\text{Hence, } \tan^{-1}(\alpha + i\beta) = \tfrac{1}{2} \tan^{-1} \frac{2\alpha}{1 - \alpha^2 - \beta^2}$$

$$+ \tfrac{1}{4}\, i \log \frac{(1 + \beta)^2 + \alpha^2}{(1 - \beta)^2 + \alpha^2}.$$

Example 2. Prove that

$$\tan^{-1}(e^{i\theta}) = \frac{n\pi}{2} + \frac{\pi}{2} - \frac{i}{2} \log \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

Let $\tan^{-1}(e^{i\theta}) = x + iy$

$$\therefore \quad \tan(x + iy) = e^{i\theta} = \cos\theta + i \sin\theta$$

hence, $\tan(x - iy) = \cos\theta - i\sin\theta$

$$\therefore \quad \tan(2x) = \tan[(x + iy) + (x - iy)]$$

$$= \frac{(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)}{1 - (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}.$$

$$= \frac{2\cos\theta}{1-1} = \infty$$

$$\therefore \quad 2x = n\pi + \frac{\pi}{2}$$

$$\therefore \quad x = \frac{n\pi}{2} + \frac{\pi}{4}$$

and $\tan 2iy = \tan[(x+iy)-(x-iy)]$

$$= \frac{2i\sin\theta}{2}$$

$$\therefore \quad \tanh 2y = \sin\theta$$

$$2y = \tanh^{-1}(\sin\theta)$$

$$= \tfrac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}, \quad \text{if } \phi = \frac{\pi}{2}-\theta$$

$$= \tfrac{1}{2}\log\frac{1+\cos\phi}{1-\cos\phi} = \tfrac{1}{2}\log\frac{2\cos^2\frac{\phi}{2}}{2\sin^2\frac{\phi}{2}}$$

$$= \tfrac{1}{2}\,2\log\cot\frac{\phi}{2} = \log\cot\frac{\phi}{2}$$

$$= -\log\tan\frac{\phi}{2} = -\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\therefore \quad y = -\frac{1}{2}\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\therefore \quad \tan^{-1}(e^{i\theta}) = \frac{n\pi}{2}+\frac{\pi}{4}-\frac{i}{2}\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

Example 3. If $\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha$, prove that

$$e^{2\phi} = \cot\frac{\alpha}{2} \text{ and that } 2\theta = n\pi + \frac{\pi}{2} + \alpha$$

$$\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha$$

$$\therefore \quad \tan(\theta-i\phi) = \tan\alpha - i\sec\alpha$$

$$\tan 2\theta = \tan[(\theta+i\phi)+(\theta-i\phi)]$$

$$= \frac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{2\tan\alpha}{1-(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}$$

$$= \frac{2\tan\alpha}{1-\tan^2\alpha-\sec^2\alpha} = \frac{2\tan\alpha}{-2\tan^2\alpha} = -\cot\alpha$$

$$= \tan\left(\frac{\pi}{2}+\alpha\right)$$

$$\therefore\ 2\theta = n\pi + \frac{\pi}{2} + \propto \text{ as required.}$$

and $\tan(2i\phi) = \tan[(\theta + i\phi) - (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$\therefore\ i\tanh(2\phi) = \frac{2\,i\sec\propto}{1 + \tan^2\propto + \sec^2\propto} = \frac{2\,i\sec\propto}{2\sec^2\propto}$$

$$\therefore\ \tanh 2\phi = \cos\propto$$

$$\therefore\ 2\phi = \tanh^{-1}(\cos\propto)$$

$$= \tfrac{1}{2}\log\frac{1+\cos\propto}{1-\cos\propto}\left[\text{from } \tanh^{-1}x = \tfrac{1}{2}\log\frac{1+x}{1-x}\right]$$

$$= \tfrac{1}{2}\log\frac{2\cos^2\frac{\propto}{2}}{2\sin^2\frac{\propto}{2}} = \log\cot\frac{\propto}{2}$$

$$\therefore\ e^{2\phi} = \cot\frac{\propto}{2}$$

Solved Problems :—

Example 1. Prove $\tanh^{-1}x = \sinh^{-1}\dfrac{x}{\sqrt{1-x^2}}$

Let $\tanh^{-1}x = y$

$$\therefore\ x = \tanh y$$

But $1 - \tanh^2 y = \text{sech}^2 y$

$$\therefore\ \text{secc}^2 y = 1 - x^2$$

$$\therefore\ \cosh^2 y = \frac{1}{1-x^2}$$

$$\therefore\ \cosh y = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore\ \sinh y = \tanh y \times \cosh y = \frac{x}{\sqrt{1-x^2}}$$

$$\therefore\ y = \tanh^{-1}x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$$

Example 2. Prove that

$$\left\{\sin(\alpha + \theta) - e^{i\alpha}\sin\theta\right\}^n = \sin^n\alpha\, e^{-in\theta}.$$

$$\text{L.H.S.} = \{\sin(\alpha+\theta - (\cos\alpha + i\sin\alpha)\sin\theta\}^n$$
$$= \{\sin\alpha\cos\theta + \cos\alpha\sin\theta - \cos\alpha\sin\theta - i\sin\alpha\sin\theta\}^n$$
$$= \sin^n\alpha\{\cos\theta - i\sin\theta\}^n$$
$$= \sin^n\alpha\{e^{-i\theta}\}^n = \sin^n\alpha . e^{-in\theta}$$

Example 3. If $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, prove that $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$.

$$\tanh\frac{u}{2} = \frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \frac{e^u-1}{e^u+1}$$

$$\text{But } u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\therefore \quad e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\therefore \tanh\frac{u}{2} = \frac{e^u-1}{e^u+1}$$

$$= \frac{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - \tan\frac{\pi}{4}}{1+\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\tan\frac{\pi}{4}} \left(\text{as } \tan\frac{\pi}{4} = 1\right)$$

$$= \tan\frac{\theta}{2} \quad \left[\frac{\tan A - \tan B}{1+\tan A\tan B} = \tan(A-B)\right]$$

Example 4. Prove that

$$(1-e^{i\theta})^{-1/2} + (1-e^{-i\theta})^{-1/2} = \left(1+\operatorname{cosec}\frac{\theta}{2}\right)^{1/2}$$

$$1-e^{i\theta} = 1-\cos\theta - i\sin\theta$$

$$= 2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\sin\frac{\theta}{2}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]$$

$$= 2\sin\frac{\theta}{2}\left[\cos\phi - i\sin\phi\right]$$

$\left\{ \text{but } \phi = \frac{\pi}{2} - \frac{\theta}{2} \right.$ to express it in the form $\cos A \quad i\sin A$

$$\therefore \left(1-e^{i\theta}\right)^{-1/2} = \left(2\sin\frac{\theta}{2}\right)^{-1/2}\left[\cos\phi - i\sin\phi\right]^{-1/2}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-1/2}\left[\cos\left(-\frac{\phi}{2}\right) - i\sin\left(-\frac{\phi}{2}\right)\right]$$

by De Moivres' Theorem

$$= \left(2\sin\frac{\theta}{2}\right)^{-1/2}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right]$$

$$\text{and } 1 - e^{-i\theta} = 1 - \cos\theta + i\sin\theta$$

$$= 2\sin\frac{\theta}{2}\left[\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right]$$

$$= 2\sin\frac{\theta}{2}\left[\cos\phi + i\sin\phi\right] \text{ where } \phi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\therefore \left(1 - e^{-i\theta}\right)^{1/2} = \left(2\sin\frac{\theta}{2}\right)^{-1/2}\left[\cos\phi + i\sin\phi\right]^{-1/2}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-1/2}\left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right]$$

$$\therefore \left(1 - e^{i\theta}\right)^{-\frac{1}{2}} + \left(1 - e^{-i\theta}\right)^{-1/2}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-1/2}\left[\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right) + \left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)\right]$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-1/2}\left(2\cos\frac{\phi}{2}\right)$$

$$= \left\{\frac{4\cos^2\dfrac{\phi}{2}}{2\sin\dfrac{\theta}{2}}\right\}^{1/2}$$

$$= \left\{\frac{1 + \cos\phi}{\sin\dfrac{\theta}{2}}\right\}^{1/2}, \quad \text{but } \phi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$= \left\{\frac{1 + \sin\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}\right\}^{1/2} = \left\{1 + \operatorname{cosec}\frac{\theta}{2}\right\}^{1/2}.$$

Example 5. If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that
$$\cos^2\theta = \pm\sin\alpha$$

We have
$$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \sin(\theta + i\phi)$$
$$= \cos\alpha + i\sin\alpha$$

Equating the real and imaginary parts, we get

$$\left.\begin{aligned} \sin\theta\cosh\phi &= \cos\alpha \\ \text{and } \cos\theta\sinh\phi &= \sin\alpha \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

To prove the required results, eliminate ϕ from (1) by using $\cosh^2\phi - \sinh^2\phi = 1$. We get

$$\frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta} = 1$$

i. e. $$\frac{1-\sin^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta} = 1$$

$$\therefore \quad \frac{1}{\sin^2\theta} - 1 = \sin^2\alpha\left\{\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\right\}$$

$$\therefore \quad \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\alpha}{\sin^2\theta\cos^2\theta}$$

$$\therefore \quad \cos^4\theta = \sin^2\alpha \qquad \therefore \quad \cos^2\theta = \pm\sin\alpha$$

Example 6. Separate into real and imaginary parts the expression

$$\sin^{-1}(e^{i\theta}).$$

Let $\sin^{-1}(e^{i\theta}) = x + iy$

$$\therefore \sin(x+iy) = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta + i\sin\theta = \sin(x+iy)$$
$$= \sin x\cosh y + i\cos x\sinh y.$$

Equating real and imaginary parts, we get

$$\cos\theta = \sin x\cosh y \qquad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\sin\theta = \cos x\sinh y \qquad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Squaring and adding (i. e. to eliminate θ), we get

$$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$$
$$= \sin^2 x(1+\sinh^2 y) + \cos^2 x\sinh^2 y$$
$$= \sin^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y$$

$$\therefore \quad \cos^2 x = \sinh^2 y$$

$$\therefore \quad \cos x = \sinh y$$

But $\cos x\sinh y = \sin\theta$ .. .. (iii)

$$\therefore \quad \cos^2 x = \sin\theta \text{ i. e. } \cos x = \sqrt{\sin\theta}$$

$$\therefore \quad x = \cos^{-1}(\sqrt{\sin\theta}) \quad \therefore \quad \therefore \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

and putting $\cos x = \sinh y$ in (iii), we get

$$\sinh^2 y = \sin\theta, \text{ i. e. } \sinh y = \sqrt{\sin\theta}$$

$$\therefore \quad y = \sinh^{-1}\sqrt{\sin\theta} = \log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta}) \quad \ldots \quad \ldots \quad \text{(v)}$$

Hence from (i) and (v), we have

$$\sin^{-1}(e^{i\theta}) = \cos^{-1}\sqrt{\sin\theta} + i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$$

Example 7. If $\cos(\theta + i\phi) = R(\cos\alpha + i\sin\alpha)$, prove that

$$\phi = \tfrac{1}{2}\log\left\{\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right\}.$$

We have

$$R\cos\alpha + iR\sin\alpha = \cos(\theta + i\phi)$$
$$= \cos\theta\cosh\phi - i\sin\theta\sinh\phi$$

Equating real and imaginary parts, we have

$$R\cos\alpha = \cos\theta\cosh\phi \quad \ldots \quad (i)$$
$$R\sin\alpha = -\sin\theta\sinh\phi \quad \ldots \quad (ii)$$

To prove the required results eliminate R from (i) and (ii) by taking the ratio, we get

$$\frac{\cos\alpha}{\sin\alpha} = -\frac{\cos\theta\cosh\phi}{\sin\theta\sinh\phi}$$

$$\therefore \tanh\phi = -\frac{\sin\alpha\cos\theta}{\sin\theta\cos\alpha}$$

$$\therefore \phi = \tanh^{-1}\left\{-\frac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta}\right\}$$

$$= \tfrac{1}{2}\log\left\{\frac{1-\dfrac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta}}{1+\dfrac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta}}\right\}$$

$$= \tfrac{1}{2}\log\left\{\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right\}$$

Example 8. Prove that

$$\frac{1+\cos\alpha+i\sin\alpha}{1-\cos\alpha+i\sin\alpha} = \cot\frac{\alpha}{2}\cdot e^{i\left(\alpha-\frac{\pi}{2}\right)}$$

$$N = 1+\cos\alpha+i\sin\alpha = 2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]$$

$$= 2\cos\frac{\alpha}{2}\cdot e^{i\alpha/2}$$

$$D = 1-\cos\alpha+i\sin\alpha = 2\sin\frac{\alpha}{2}\left[\sin\frac{\alpha}{2}+i\cos\frac{\alpha}{2}\right]$$

$$= 2\sin\frac{\alpha}{2}\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right]$$

$$= 2\sin\frac{\alpha}{2}\cdot e^{i\left(\alpha-\frac{\pi}{2}\right)}$$

Hence

$$\text{L. H. S.} = \frac{1 + \cos\alpha + i\sin\alpha}{1 - \cos\alpha + i\sin\alpha} = \frac{2\cos\frac{\alpha}{2}\, e^{i\alpha/2}}{2\sin\frac{\alpha}{2} \cdot e^{i\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)}}$$

$$= \cot\frac{\alpha}{2} \cdot e^{i\left[\frac{\alpha}{2} - \left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]}$$

$$= \cot\frac{\alpha}{2} \cdot e^{i\left(x - \frac{\pi}{2}\right)}$$

Examples : II—B

(1) Prove that

(i) $\sin 2\alpha + i\sinh 2\beta = 2\sin(\alpha + i\beta)\cos(\alpha - i\beta)$

(ii) $\cos(\alpha + i\beta) + i\sin(\alpha + i\beta) = e^{-\beta}(\cos\alpha + i\sin\alpha)$

(iii) $\sin(\alpha + n\theta) - e^{i\alpha}\sin n\theta = e^{in\theta}\sin\alpha$

(2) Prove that $\sinh^{-1} x = \cosh^{-1}\sqrt{1+x^2}$.

(3) If $\tan\alpha = \tan x\tanh y$ and $\tan\beta = \cot x\tanh y$, show that

$$\tan(\alpha + \beta) = \sinh 2y\,\text{cosec}\,2x.$$

(4) Separate into real and imaginary parts

(i) $\cot(x + iy)$ (ii) $\sec(x + iy)$ (iii) $\tanh(x + iy)$

(iv) $\text{sech}(x + iy)$

[Ans. (i) $\dfrac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}$ (ii) $2\left(\dfrac{\cos x\cosh y + i\sin x\sinh y}{\cos 2x + \cosh 2y}\right)$

(iii) $\dfrac{\sinh 2x + i\sin 2y}{\cosh 2x + \cos 2y}$ (iv) $2\left(\dfrac{\cosh x\cos y - i\sinh x\sin y}{\cosh 2x + \cos 2y}\right)$.

(5) If $\tan(x + iy) = i$ where x and y are real, prove that x is indeterminate and y is infinite.

(6) If $y = \log\tan x$, prove that

(i) $\sinh ny = \frac{1}{2}(\tan^n x - \cot^n x)$

(ii) $2\cosh ny\,\text{cosec}\,2x = \cosh(n+1)y + \cosh(n-1)y$

(7) Show that

$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

(8) If $x + iy = \sinh(3 + 4i)$, find numerical values of x and y three places of decimals.

[Ans. $x = -6.546$, $y = -7.6$

(9) Show that all the solutions of the equation

$$\sin z = 2i \cos z$$

are given by $z = \frac{n\pi}{2} + \frac{i}{2} \log 3$, where n is zero or positive integer.

(10) Find all complex numbers which satisfy the following equations

(i) $\cosh z = -1$ and $|z| < 5$

(ii) $\tan z = \frac{1}{2}(1 - i)$

[Ans. (i) $\pm i\pi$ (ii) $z = n\pi + \tan^{-1} 2 - \frac{1}{4} i \log 5$]

(11) Find in the form $a + ib$, the expression $\cos^{-1}\left(\frac{3i}{4}\right)$.

$$\left[\text{Ans. } \frac{\pi}{2} - i \log 2\right]$$

(12) If $\sin(\alpha + i\beta) = x + iy$, prove that

$$\frac{x^2}{\cosh^2 \beta} + \frac{y^2}{\sinh^2 \beta} = 1 \text{ and } \frac{x^2}{\sin^2 \alpha} - \frac{y^2}{\cos^2 \alpha} = 1.$$

(13) If $\cos\left(\frac{\pi}{4} + ia\right) \cosh\left(b + i\frac{\pi}{4}\right) = 1$, a and b being real, show that

$$2b = \pm \log(2 + \sqrt{3}).$$

(14) Prove that $\tan \frac{u + iv}{2} = \frac{\sin u + i \sinh v}{\cos u + \cosh v}$.

(15) If $\tan(\theta + i\phi) = \cos \alpha + i \sin \alpha$, prove that

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4} \text{ and } \phi = \frac{1}{2} \log \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

(16) $\tan(x + iy) = \sin(u + iv)$, prove that

$$\frac{\tan u}{\tanh v} = \frac{\sin 2x}{\sinh y}$$

(17) If $\tan(\alpha + i\beta) = x + iy$, show that

$x^2 + y^2 + 2x \cot 2\alpha = 1$ and $x^2 + y^2 - 2y \coth 2\beta + 1 = 0$.

(18) If $\alpha + i\beta = c \tan(x + iy)$, then $\tan 2x = \frac{2c\alpha}{c^2 - \alpha^2 - \beta^2}$.

(19) If $\operatorname{cosec}\left(\frac{\pi}{4} + ix\right) = u + iv$, where x, u, v are all real, show that $(u^2 + v^2)^2 = 2(u^2 - v^2)$.

(20) If $\sin(\alpha + i\beta) = R(\cos\theta + i \sin\theta)$, prove that

$R^2 = \frac{1}{2}[\cosh 2\beta - \cos 2\alpha]$ and $\tan\theta = \tanh\beta \cot\alpha$.

(21) If $\sinh(\theta + i\phi) = e^{i\alpha}$ prove that

$$\sinh^4\theta = \cos^2\alpha = \cos^4\phi.$$

(22) If p, q be the imaginary cube roots of unity, prove that

$$pe^{px} + qe^{qx} = -e^{-x/2}\left\{\sqrt{3}\sin\frac{\sqrt{3}}{2}x + \cos\frac{\sqrt{3}}{2}x\right\}.$$

(23) The complex numbers $e^{i\alpha}$, $e^{i\beta}$ are represented on Argand, diagram by vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively. If OP=OA+OB, prove that OP represents complex number $2e^{i\left(\frac{\alpha+\beta}{2}\right)}\cos\left(\frac{\alpha-\beta}{2}\right)$

(24) If $\alpha = 1 + i$, $\beta = 1 - i$ and $\cot\phi = x + 1$ prove that

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \sin(n\phi)\operatorname{cosec}^n\phi.$$

2·15 Logarithms of a complex quantity :—

Let $z = x + iy$

Expressing the complex number in general polar form, we have

$$z = x + iy = r\,[\cos(2\pi k + \theta) + i\sin(2\pi k + \theta)]$$

where $r = \sqrt{x^2 + y^2}$

and $\theta = \tan^{-1}\dfrac{y}{x}$

$$= re^{i(2\pi k + \theta)} \quad [\text{from result (12)}]$$

$$\therefore \operatorname{Log} z = \log r + i\,(2\pi k + \theta)$$

This shows that the *logarithm of a complex quantity* $x + iy$ is *multivalued* (for different values of k) and is written as Log $(x + iy)$ and hence

$$\boxed{\operatorname{Log}(x+iy) = \log\sqrt{x^2+y^2} + i\left(2\pi k + \tan^{-1}\frac{y}{x}\right)} \quad \ldots (20)$$

If we put k equal to zero in the value of Log $(x + iy)$ the result is called the *principal value of the logarithm* and is denoted by

$$\boxed{\begin{aligned}&\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)\\ &\text{or } \log(x+iy) = \log r + i\,\theta\end{aligned}} \quad \ldots (21)$$

This distinction between log and Log is to be clearly understood.

Solved Problems :—

Example 1. Find the values of Log (-5) and log $(1+i)$.

Log (-5) :—

$-5 = 5(\cos\pi + i\sin\pi)$. Hence $r = 5, \theta = \pi$

$\therefore$ Log $(-5) = \log 5 + i(2\pi k + \pi)$ [by result (20)]

log $(1+i)$:—

$$1 + i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

Hence $r = \sqrt{2}$ and $\theta = \dfrac{\pi}{4}$

$$\therefore \quad \log(1+i) = \tfrac{1}{2}\log 2 + i\,\frac{\pi}{4}$$

Example 2. Prove the following :—

(i) $\log\tan\left(\dfrac{\pi}{4} + i\,\dfrac{x}{2}\right) = i\tan^{-1}(\sinh x)$

(ii) $\log\dfrac{a+ib}{a-ib} = 2\,i\tan^{-1}\dfrac{b}{a}$

(i) $$\tan\left(\frac{\pi}{4} + i\,\frac{x}{2}\right) = \frac{\sin 2\dfrac{\pi}{4} + i\sinh 2\dfrac{x}{2}}{\cos 2\dfrac{\pi}{4} + \cosh 2\dfrac{x}{2}} \quad \left[\begin{array}{c}\text{using the}\\ \text{results } (b) \text{ of}\\ \text{art. } 2\cdot 14.\end{array}\right]$$

$$= \frac{1 + i\sinh x}{\cosh x}$$

$$= \frac{1}{\cosh x} + i\,\frac{\sinh x}{\cosh x}$$

$$= r[\cos\theta + i\sin\theta]$$

$$\therefore \quad r\cos\theta = \frac{1}{\cosh x}\ ; \ r\sin\theta = \frac{\sinh x}{\cosh x}$$

$$\therefore \quad r^2 = \frac{1 + \sinh^2 x}{\cosh^2 x} = \frac{\cosh^2 x}{\cosh^2 x} = 1$$

and $\tan\theta = \sinh x$ i.e. $\theta = \tan^{-1}(\sinh x)$

Hence using the result (21)

$$\log \tan\left(\frac{\pi}{4} + i\frac{x}{2}\right) = \log 1 + i \tan^{-1} (\sinh x)$$

$$= i \tan^{-1} (\sinh x).$$

(ii) If $a + ib = r(\cos\theta + i\sin\theta)$

we have $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}\dfrac{b}{a}$

and

$$a - ib = r(\cos\phi + i\sin\phi)$$

where $r = \sqrt{a^2 + b^2}$, $\phi = -\tan^{-1}\dfrac{b}{a}$

$$\therefore \quad \log\frac{a+ib}{a-ib} = \log(a+ib) - \log(a - b)$$

$$= \left\{\tfrac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a}\right\}$$

$$- \left\{\tfrac{1}{2}\log(a^2+b^2) - i\tan^{-1}\frac{b}{a}\right\}$$

$$= 2i\tan^{-1}\frac{b}{a}.$$

Example 3. Prove that $i\log\dfrac{x-i}{x+} = \pi - 2\tan^{-1}x.$

Let $x + i = re^{i\theta}$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{x}\right).$$

and $x - i = re^{-i\theta}$.

$$\therefore \quad i\log\frac{x-i}{x+i} = i\log\left(e^{-2i\theta}\right)$$

$$= i(-2i\theta) = 2\theta \quad \ldots\ldots \quad (i)$$

But $\tan\theta = \dfrac{1}{x}$ $\therefore$ $x = \cot\theta = \tan\left(\dfrac{\pi}{2} - \theta\right).$

$$\therefore \quad \frac{\pi}{2} - \theta = \tan^{-1}x \text{ and } \theta = \frac{\pi}{2} - \tan^{-1}x.$$

Puting this values of θ in (i)

$$i\log\frac{x-i}{x+i} = \pi - 2\tan^{-1}x.$$

In problems involving separation of real and imaginary parts of the expression $(\alpha + i\beta)^{x+iy}$, express the quantity in terms of base e and proceed thus :—

Let $(\alpha + i\beta) = r[\cos(2\pi k + \theta) + i \sin(2\pi k + \theta)]$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1} \dfrac{\beta}{\alpha}$

$$\therefore \quad (\alpha + i\beta)^{(x+iy)} = e^{(x+iy)\,\mathrm{Log}\,(\alpha + i\beta)}$$

$$[\text{using } a^x = e^{x \log_e a}]$$

$$= e^{(x+iy)\,[\log r + i(2\pi k + \theta)]}$$

$$= e^{[x \log r - y(2\pi k + \theta)]} \times e^{i\{y \log r + x(2\pi k + \theta)\}}$$

$$[\text{as } e^{A+B} = e^A e^B]$$

$$= e^{[x \log r - y(2\pi k + \theta)]} \times \{\cos[y \log r + x(2\pi k + \theta)] + i \sin[y \log r + x(2\pi k + \theta)]\}$$

which gives the real and imaginary parts of $(\alpha + i\beta)^{x+iy}$

Solved Problems :–

Example 1. Prove that

$$i^a = \cos\{(2m + \tfrac{1}{2})\pi a\} + i \sin\{(2m + \tfrac{1}{2})\pi a\}$$

$$i^a = e^{a\,\mathrm{Log}\,i}$$

$$= e^{a[\log 1 + i(2m + \frac{1}{2})\pi]} \quad \left[\text{as } i = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right]$$

$$= e^{i(2m + \frac{1}{2})\pi a}$$

$$= \cos\{(2m + \tfrac{1}{2})\pi a\} + i \sin\{(2m + \tfrac{1}{2})\pi a\}.$$

In case of problems of identities, take logarithms of both the sides and separate them into real and imaginary parts and by equating the real and imaginary parts of both the sides, the required result can be proved. This method will be clear from the following problems.

Example 2. Prove that the real part of the principal value of $(i)^{\log(1+i)}$ is $e^{-\frac{\pi^2}{8}} \cos\left(\dfrac{\pi}{4} \log 2\right)$.

$$(i)^{\log(1+i)} = e^{\log(1+i)\,\log i}$$

$$= e^{\left[\frac{1}{2}\log 2 + i\frac{\pi}{4}\right]\left[\log 1 + i\frac{\pi}{2}\right]} \quad (\log 1 = 0)$$

$$= e^{\left\{i \frac{\pi}{4} \log 2 - \frac{\pi^2}{8}\right\}}$$

$$= \left(e^{-\frac{\pi^2}{8}}\right)\left(e^{i\frac{\pi}{4}\log 2}\right)$$

$$= e^{-\pi^2/8}\left\{\cos\left(\frac{\pi}{4}\log 2\right) + i \sin\left(\frac{\pi}{4}\log 2\right)\right\}$$

Hence the result.

Example 3. If $i^{(\alpha+i\beta)} = \alpha + i\beta$, prove that
$\alpha^2+\beta^2 = e^{-(4m+1)\pi\beta}$.

Take the logarithm of both sides of $i^{(\alpha+i\beta)} = \alpha + i\beta$, to the base e.

We get

$$(\alpha+i\beta) \text{ Log } i = \text{Log } (\alpha+i\beta)$$

i. e. $(\alpha+i\beta)\,[\,i\,(2m+\tfrac{1}{2})\,\pi\,] = \tfrac{1}{2}\log(\alpha^2+\beta^2)$

$$+\, i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$

i. e. $[-(2m+\tfrac{1}{2})\pi\beta + i(2m+\tfrac{1}{2})\pi\alpha]$

$$= \tfrac{1}{2}\log(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$

Equating the real parts, we have

$$-(2m+\tfrac{1}{2})\pi\beta = \tfrac{1}{2}\log(\alpha^2+\beta^2)$$

$$\therefore \quad \log(\alpha^2+\beta^2) = -(4m+1)\pi\beta$$

$$\therefore \quad \alpha^2+\beta^2 = e^{-(4m+1)\pi\beta}.$$

Example 4. If $(x + ib)^p = m^{(x+iy)}$, prove that one of the values of $\frac{y}{x}$ is $\dfrac{2\tan^{-1}\frac{b}{a}}{\log(a^2+b^2)}$.

Taking logarithms of both the sides of the identity and considering only the *principal* values, we have

$$p \log(a + ib) = (x + iy) \log m$$

i. e. $p\left\{\tfrac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a}\right\} = (x+iy)\{\log m + i\,0\}$

Hence equating the real and imaginary parts,

$$\frac{p}{2}\log(a^2+b^2) = x\log m \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

$$p\tan^{-1}\frac{b}{a} = y\log m \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

$\therefore$ Dividing (ii) by (i), we get

$$\frac{y}{x} = \frac{2\tan^{-1}\frac{b}{a}}{\log(a^2 + b^2)}.$$

Example 5. Find the general value of $\text{Log}_{(-3)}(-2)$

Let $\text{Log}_{(-3)}(-2) = x + iy$

Changing the base of the logarithms, we get

$$\text{Log}_e(-2) = (x + iy)\,\text{Log}_e(-3)$$

i. e. $\therefore$ $\log 2 + i(2n\pi + \pi) = (x + iy)$

$$[\log 3 + i(2m\pi + \pi)]$$

$$= \{x\log 3 - (2m + 1)\pi y\} + i\{(2m + 1)\pi x + y\log 3\}$$

Equating the real and imaginary parts, we get

$$x\log 3 - (2m + 1)\pi y = \log 2 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$(2m + 1)\pi x + y\log 3 = (2n + 1)\pi \quad \ldots \quad \ldots \quad \text{(ii)}$$

Solving the linear simultaneous (i), (ii) we get

$$x = \frac{(2m + 1)(2n + 1)\pi^2 + (\log 2)(\log 3)}{(\log 3)^2 + (2m + 1)^2\pi^2}$$

and $$y = \frac{(\log 3)(2n + 1)\pi - (2m + 1)\pi\log 2}{(\log 3)^2 + (2m + 1)^2\pi^2}$$

Examples : II – C

(1) Find in the form $a + ib$ (consider only principal values)

(i) $\log\dfrac{3 - i}{3 + i}$ (ii) $(1 + i)^i$ (iii) i^i (iv) $(1 - i)^{(1 - i)}$

(v) $2^{1 + i}$ (vi) $(-i)^{-(1 - i)}$.

Ans. :– (i) $-0.6434i$ (ii) $e^{-\pi/4}[\cos(\frac{1}{2}\log 2) + i\sin(\frac{1}{2}\log 2)]$

(iii) $e^{-\pi/2}$

(iv) $e^{\left(\frac{1}{2}\log 2 - \frac{\pi}{4}\right)}[\cos\theta - i\sin\theta]$, where $\theta = \frac{1}{2}\log 2 + \frac{\pi}{4}$

(v) $e^{\log 2}[\cos(\log 2) + i\sin(\log 2)]$

(vi) $ie^{\pi/2}$

Prove that :–

(2) $a^i = e^{-2n\pi}\{\cos(\log a) + i\sin(\log a)\}$.

(3) $\log(1 + i\tan\alpha) = \log\sec\alpha + i\alpha$.

(4) $\log(1 + \cos 2\theta + i\sin 2\theta) = \log(2\cos\theta) + i\theta$.

(5) $\log\left(\dfrac{1}{1-e^{i\theta}}\right) = \log\left(\tfrac{1}{2}\operatorname{cosec}\dfrac{\theta}{2}\right) + i\left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right).$

(6) $\log \dfrac{\cos(x-iy)}{\cos(x+iy)} = 2i \tan^{-1}(\tan x \tanh y).$

(7) $\log \dfrac{\sin(x+iy)}{\sin(x-iy)} = 2i \tan^{-1}(\cot x \tanh y).$

(8) $\log i = i\,\dfrac{\pi}{2}.$

(9) Prove that $i^{i^{i}} = \cos\theta + i\sin\theta.$

where $\theta = (2n+\tfrac{1}{2})\dfrac{\pi}{2}\; e^{-(2m+\frac{1}{2})\pi}$

(10) If $\dfrac{(1+i)^{x+iy}}{(1-i)^{x-iy}} = \alpha + i\beta$, prove that one value of $\tan^{-1}\dfrac{\beta}{\alpha}$

is $\tfrac{1}{2}\pi x + y \log 2.$

(11) Show that $\operatorname{Log}_i i = \dfrac{4n+1}{4m+1}.$

(12) Prove that the general value of $\operatorname{Log}_2(-3)$ is

$$\frac{\{\log 3 \log 2 + 2n(2m+1)\pi^2\} + i\pi\{(2m+1)\log 2 - 2n\log 3\}}{(\log 2)^2 + 4n^2\pi^2}$$

and its principal value is given by

$$\frac{\log 3 + \pi i}{\log 2}.$$

(13) Find the principal value of $(1+2i)^{2+3i}$.

[Ans. $0.0150 - 0.1786\,i$].

(14) Solve for z if $e^z = 1 + i\sqrt{3}$.

$\left[\text{Ans. } z = \log 2 + i\left(2n\pi + \dfrac{\pi}{3}\right)\right].$

2.16. j (= i) as an operator (Electrical circuits) :–

We have

$$ai = ae^{i\pi/2} = ia$$

$$ai^2 = ae^{i\pi} = -a$$

$$ai^3 = ae^{i\,3\pi/2} = -ia$$

$$ai^4 = ae^{i\,(2\pi)} = a$$

Thus if we take a radius vector of length 'a' along a horizontal line, the effect of raising i to a power n is equivalent to turning this initial radius vector through an angle $n\,\dfrac{\pi}{2}$.

(*i*) *Operation of* j $(= i)$ *on* $a \sin pt$:—

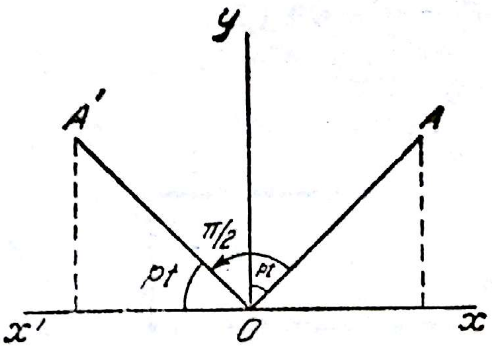

Fig. 14

In Fig. 14, $a \sin pt$ is the projection of vector $\overrightarrow{OA}$ $(= a)$ on the horizontal line, where pt is the angle made by it with vertical. Then $j\ (a \sin pt)$ represents the projection of the vector $\overrightarrow{OA'}$ $(= a)$ on the horizontal line, when $\overrightarrow{OA}$ is turned through $\dfrac{\pi}{2}$.

$\therefore\ j\,(a \sin pt) =$ Projection of $\overrightarrow{OA'}$ on $xox' = a \cos pt$

$$\boxed{j\,(a \sin pt) = a \cos pt} \qquad \ldots \quad \ldots \qquad (22)$$

(*ii*) *Operation of* $(a + jb)$ *on* $\sin pt$:—

$$(a + jb) \sin pt = a \sin pt + jb \sin pt$$

$$= a \sin pt + b \cos pt \ [\text{from (i)}]$$

$$\therefore \quad \boxed{(a + jb) \sin pt = \sqrt{a^2 + b^2} \sin (pt + \alpha)}$$

$$\text{where } \tan \alpha = \frac{b}{a} \qquad \ldots \quad \ldots \qquad (23)$$

Operatian of $(a - jb)$ *on* $\sin pt$:—

$$(a - jb) \sin pt = a \sin pt - jb \sin pt$$

$$= a \sin pt - b \cos pt$$

$$\therefore \quad \boxed{(a - jb) \sin pt = \sqrt{a^2 + b^2} \sin (pt - \alpha)}$$

$$\text{where } \tan \alpha = \frac{b}{a} \quad \ldots \quad \ldots \quad (24)$$

(iii) *Operation of* $\frac{1}{a + jb}$ *on* $\sin pt$:—

$$\frac{1}{a + jb} \sin pt = \frac{a - jb}{a^2 + b^2} \sin pt$$

$$= \frac{1}{a^2 + b^2} \sqrt{a^2 + b^2} \sin (pt - \alpha),$$

$$\text{where } \tan \alpha = \frac{b}{a} \text{ [from (iii)]}$$

$$\therefore \quad \boxed{\frac{1}{a + jb} \sin pt = \frac{1}{\sqrt{a^2+b^2}} \sin (pt - \alpha)} \quad \ldots \quad (25)$$

Similarly,

$$\therefore \quad \boxed{\frac{1}{a - jb} \sin pt = \frac{1}{\sqrt{a^2+b^2}} \sin (pt + \alpha)} \quad \ldots \quad (26)$$

$$\text{where } \tan \alpha = \frac{b}{a}$$

In an electrical circuit containing a resistance R, inductance L and capacity C in series, we know that if a current I flows through the circuit at any time, due to the applied harmonic E. M. F. $E_0 \sin pt$, we have

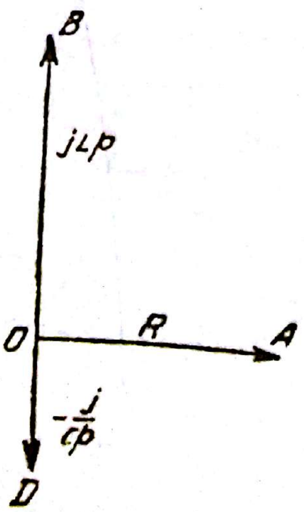

Fig. 15.

E_R = RI in phase with I

E_L = LpI in quadrature with I (leading)

$E_C = \frac{I}{Cp}$ in quadrature with I (lagging)

where E_R, E_L and E_C are voltage drops across R, L and C respectively.

This is diagramatically represented in the adjoining figure.

The total impedence in the circuit is given by addition of these vectors. Thus if Z represents the total impedence

$$Z = R + j\left(Lp - \frac{1}{Cp}\right).$$

Hence, if $E_o \sin pt$ be applied voltage, the current I in the circuit is given by

$$I = \frac{1}{R + j\left(Lp - \frac{1}{Cp}\right)} E_o \sin pt$$

$$= \frac{E_o}{\sqrt{R^2 + \left(Lp - \frac{1}{Cp}\right)^2}} \sin (pt - \alpha) \quad \text{[from (25)]}$$

$$\text{where } \alpha = \tan^{-1} \frac{\left(Lp - \frac{1}{Cp}\right)}{R}.$$

Exmples : II — D

1. On the Argand diagram the centre of a regular hexagon represents the number $2 - i$ and one vertex represents the number $-1 + i$ Find the numbers represented by the remaining vertices.

   [Ans. $2 - i + (-3+2i)\left(\cos \frac{r\pi}{3} + i \sin \frac{r\pi}{3}\right)$ for $r = 1, 2, 3, 4, 5$].

2. If ABCDEF is a regular hexagon and A, D represent given complex numbers z and z', then the numbers represented by B, E, C, F are given by

$$\tfrac{1}{2}(z + z') + \tfrac{1}{2}(z - z')(\cos \theta + i \sin \theta)$$

where θ has values $\pm \frac{\pi}{3}$ and $\pm \frac{2\pi}{3}$.

3. If z_1, z_2 are complex numbers, find the numbers z and z' so that the points z, z', and z_1, z_2 may be the opposite corners of a square.

   [Ans. $z = \frac{1}{2}(z_1 + z_2) - \frac{i}{2}(z_1 - z_2)$; $z' = \frac{1}{2}(z_1 + z_2) + \frac{i}{2}(z_1 - z_2)$]

4. The points A, B, C, D in Argand's diagram represent complex numbers $9 + i$, $4 + 13i$, $-8 + 8i$ and $-3 - 4i$. Prove that ABCD is a square.

5. If z_1, z_2, z_3 represent vertices of an equilateral triangle, prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

6. If z_1, z_2, z_3, z_4 are vertices of a parallelogram, then prove
$$z_1 - z_2 + z_3 - z_4 = 0.$$

7. The vertices of a triangle ABC are $1 + 2i$, $4 - 2i$ and $1 - 6i$. Prove that the triangle is isosceles.

8. In an Argand's diagram the points A, B, C, D are respectively z^{-1}, 1, z and z^2 where z is a complex number. If $|z - 1| = 1$ show that AB is parallel to OC and BC is parallel to OD, where O is the origin.

9. Prove that the points representing three roots of the equation $z^3 = i(z - 1)^3$ in the Argand's diagram are collinear.

10. Given that $z = \sqrt{3} + i$

(a) represent z and $\dfrac{4}{z}$ as vectors on Argand's diagram,

(b) show that the sq. root of $\left(z - \dfrac{4}{z}\right)$ is $1 + i$,

(c) show that $z^n + \left(\dfrac{1}{4}z\right)^{-n} = 2^{n+1} \cos \dfrac{n\pi}{6}$.

11. A square lies entirely in the second quadrant. If one of the sides join the points -2 and $2i$, find the complex numbers representing other two vertices.

[Ans. $-2 + 4i, -4 + 2i$]

12. Given that
$$\frac{R_1 + j\omega L}{R_3} = \frac{R_2}{R_4 - \dfrac{j}{\omega C}} \quad (j^2 = -1).$$
show that
$$L = \frac{R_2 R_3 C\omega}{1 + \omega^2 R_4^2 C^2} \text{ and } R_1 = \frac{R_2 R_3 R_4 \omega^2 C^2}{1 + \omega^2 R_4^2 C^2}.$$

13. Show that the roots of the equation $(x - 1)^5 = 32(x + 1)^5$ are given by
$$x = \frac{\left(-3 + 4i \sin \dfrac{2r\pi}{5}\right)}{\left(5 - 4\cos \dfrac{2r\pi}{5}\right)}, \text{ where } r = 0, 1, 2, 3, 4.$$

14. Obtain the solutions of the equation $(x + 1)^8 + x^8 = 0$.

$$\left[\text{Ans. } x = \frac{1}{\cos \dfrac{(2k+1)\pi}{8} + i \sin \dfrac{(2k+1)\pi}{8} - 1} \text{ for } k = 0, 1, 2, \ldots, 7.\right]$$

15. Given that $1 + 2i$ is one root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find the other roots.

[Ans. $1 - 2i, \frac{1}{2}(1 \pm i\sqrt{3})$].

16. Solve the equation $x^5 - 1 = 0$, hence show that

$$x^5 - 1 = (x - 1)\left(x^2 + 2x\cos\frac{\pi}{5} + 1\right)\left(x^2 + 2x\cos\frac{3\pi}{5} + 1\right).$$

17. Solve the equation $z^2 + 2(1 + 2i)z - (11 + 2i) = 0$. where $z = x + iy$ and verify that the sum of the roots is $-2(1 + 2i)$ and product is $-(11 + 2i)$.

[Ans. $2 - i; -4 - 3i$],

18. Solve $z^2(1 - z^2) = 16$.

$$\left[\text{Ans.} \quad -\frac{3}{2} \pm \frac{\sqrt{7}}{2}i, \frac{3}{2} \pm \frac{\sqrt{7}}{2}i\right]$$

19. If $r_1 e^{i\theta_1} + r_2 e^{i\theta_2} = Re^{i\phi}$, show that

$$R = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

$$\text{and } \phi = \tan^{-1}\left[\frac{r_1 \sin\theta_1 + r_2 \sin\theta_2}{r_1 \cos\theta_1 + r_2 \cos\theta_2}\right]$$

20. Prove

$$\sum_{r=1}^{n-1} \cos\frac{2r\pi}{n} = -1$$

21. Show that all the n^{th} roots of unity are given by expressions

$$1, \lambda, \lambda^2 \ \ldots \ldots \lambda^{n-1}, \text{ where } \lambda = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \text{ and}$$

those of -1 are given by

$$\mu, \mu^3, \mu^5, \ldots \ldots, \mu^{2n-1}, \text{ where } \mu = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}.$$

22. Prove that the continued product of the three values of $i^{2/3}$ is -1.

23. Express $\sin 7\theta$ and $\cos 7\theta$ each in terms of the powers of $\cos\theta$ and $\sin\theta$. From your result express $\dfrac{\sin 7\theta}{\sin\theta}$ in powers of $\sin\theta$ only.

$$\left[\begin{array}{l} \cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta \\ \sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta \\ \dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta \end{array}\right]$$

24. Prove that :—

(i) $\tan^{-1} i\left(\dfrac{x-a}{x+a}\right) = -\dfrac{i}{2}\log\dfrac{a}{x}$.

(ii) $\operatorname{sech}^{-1}(\sin\theta) = \log\cot\dfrac{\theta}{2}$.

(iii) $\sinh x + \sin x = 2\left(x + \dfrac{x^5}{5!} + \dfrac{x^9}{9!} + \cdots\right)$.

25. If $x + iy = Ae^{int} + Be^{-int}$, where $A = a_1 + ia_2$, $B = b_1 + ib_2$ assuming that $x, y, a_1, a_2, b_1, b_2, n, t$ are real, find the values of x and y and find an equation in x and y, which does not contain t.

$$\left[\begin{array}{l}\text{Ans.}\\ x = (a_1+b_1)\cos nt - (a_2-b_2)\sin nt\\ y = (a_2+b_2)\cos nt + (a_1-b_1)\sin nt\\ x^2\{(a_1-b_1)^2 + (a_2+b_2)^2\} + y^2\{(a_1+b_1)^2 + (a_2-b_2)^2\}\\ \qquad - 4xy\,(a_1b_2+a_2b_1) = (a_1^2+a_2^2-b_1^2-b_2^2)^2\end{array}\right]$$

26. If $V = Ae^{nx} + Be^{-nx}$, where A, B are unknown constants and if $V = V_0$ when $x = 0$ and $V = 0$ when $x = l$, show that

$$V = V_0\,\frac{\sinh n(l-x)}{\sinh nl}.$$

27. Show that for any real numbers a and b

$$e^{2ai\cot^{-1} b}\left\{\frac{bi-1}{bi+1}\right\}^{-a} = 1.$$

28. If $\cos(\theta + i\phi) = \sec(\alpha + i\beta)$, prove that

$\tanh^2\phi\cosh^2\beta = \sin^2\alpha$.

$\tanh^2\beta\cosh^2\phi = \sin^2\theta$.

29. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that

$$\sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \frac{4x}{x^2+y^2}.$$

$$\sec(\alpha + i\beta) - \sec(\alpha - i\beta) = \frac{4iy}{x^2+y^2}.$$

30. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x + y + z = 0$, prove that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 0$.

31. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, show that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$$

and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$$

32. If a, b are real integers, show that

$$(re^{i\theta})^{(a+ib)} = r^a e^{-b\theta}\{\cos(a\theta+b\log r)+i\sin(a\theta+b\log r)\}$$

33. Show that the principal value of

$$\frac{(a+ib)^{p+iq}}{(a-ib)^{p-iq}}$$

is $\cos 2(p\theta+q\log r)+i\sin 2(p\theta+q\log r)$

where $\theta=\tan^{-1}\dfrac{b}{a}$ and $r=\sqrt{a^2+b^2}$.

34. If $x+iy=\cos(\alpha+i\beta)$, express x and y in terms of α, β. Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are two values of λ obtained from the equation

$$\lambda^2-(x^2+y^2+1)\lambda+x^2=0.$$

35. If $a^{\alpha+i\beta}=(x+iy)^{p+iq}$, principal values being considered; prove that

$$\alpha=\tfrac{1}{2}p\log_a(x^2+y^2)-q\tan^{-1}\frac{y}{x}\log_a e$$

and $\log_a(x^2+y^2)=\dfrac{2(\alpha p+\beta q)}{p^2+q^2}$.

36. Prove that the general value of $(1+i\tan\alpha)^{-i}$ is

$$e^{2m\pi+\alpha}\{\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)\},$$

37. Find the sum to infinite terms: —

(i) $1+\dfrac{1}{3}\cos x+\dfrac{1}{9}\cos 2x+\dfrac{2}{27}\cos 3x+\ \ldots\ \ldots$

$$\left[S=\frac{9-3\cos x}{10-6\cos x}\right].$$

(ii) $1+x\cos\theta+\dfrac{x^2}{2!}\cos 2\theta+\dfrac{x^3}{3!}\cos 3\theta+\ \ldots\ \ldots$

$$[S=e^{x\cos\theta}\cos(x\sin\theta)].$$

38. If $x+iy=\dfrac{1}{i}\log\dfrac{1+ie^{i\theta}}{1-ie^{i\theta}}$, prove that

$$x=\frac{\pi}{2}\text{ and } y=\log(\sec\theta+\tan\theta).$$

39. Prove that the each of the products $\cos(x+iy)\cos(x-iy)$ and $\sin(x+iy)\sin(x-iy)$, is real, x, y being real.

40. If $\frac{u-1}{u+1} = \sin(x+iy)$, where $u = \alpha + i\beta$, show that the amplitude of u is $\theta + \phi$, where

$$\tan\theta = \frac{\cos x \sinh y}{1+\sin x \cosh y}\ ; \ \tan\phi = \frac{\cos x \sinh y}{1-\sin x \cosh y}$$

41. If $\frac{x+iy-c}{x+iy+c} = e^{u+iv}$, where x, y, u, v and c are real, show that

$$x = -\frac{c \sinh u}{\cosh u - \cos v},\ y = \frac{c \sin v}{\cosh u - \cos v}$$

and if $x^2 + y^2 = c^2$, prove that $v = (2n+1)\frac{\pi}{2}$, where n is an integer.

42. If $x + iy = c \cot(u + iv)$, show that

$$\frac{x}{\sin 2u} = \frac{-y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}.$$

43. If $\tan\left(\frac{\pi}{4} + iv\right) = re^{i\theta}$, show that $r = 1$, $\tan\theta = \sinh 2v$ and $\tanh v = \tan\frac{\theta}{2}$.

44. If $x + iy = \tanh\left(u + i\frac{\pi}{4}\right)$, where u is real, find x, y in terms of u and show that $x^2 + y^2 = 1$.

45. If $z = x + iy$ and $Z = X + iY$, show that

if $z = \frac{Z-1}{Z+1}$, then $x^2 + y^2 = \frac{(X-1)^2 + Y^2}{(X+1)^2 + Y^2}$.

46. If $z = x + iy = r(\cos\theta + i\sin\theta)$, prove that

$$\sqrt{z} = \pm\frac{1}{\sqrt{2}}(\sqrt{r+x} + i\sqrt{r-x}) \text{ or}$$

$\pm\frac{1}{\sqrt{2}}(\sqrt{r+x} - i\sqrt{r-x})$, according as y is positive or negative.

47. If $\log\log(x+iy) = p + iq$, then prove that

$$y = x\tan\{\tan q \log\sqrt{x^2+y^2}\}.$$

48. Prove that the values of $z\,(= x + iy)$ satisfying the equation $\sin(x+iy) = 3$ are

$$n\pi + (-1)^n\left\{\frac{\pi}{2} + i\log(3 + 2\sqrt{2})\right\}$$

49. If $\tan \frac{x}{2} = \tanh \frac{u}{2}$, prove that

(i) $\sinh u = \tan x$ (ii) $\cosh u \cos x = 1$ (iii) $v = \log \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$

50. Express $\cosh (1 + i)$ in the form $a + ib$.

The current entering a telephone line is given by the real part of the expression

$$\frac{\cos \omega t + i \sin \varpi t}{\cosh (p + ip)}.$$

Express the current in the form $A \sin (\omega t + \alpha)$.

$$\left[\text{current} = \frac{\sin [\varpi t + \tan^{-1} (\cot p \coth p)]}{\sqrt{\cosh^2 p - \sin^2 p}} \right]$$

51. Considering only principal values, prove that the real part of $(1 + i\sqrt{3})^{(1 + i\sqrt{3})}$ is

$$2e^{\frac{-\pi}{3\sqrt{3}}} \cos \left(\frac{\pi}{3} + \sqrt{3} \log 2 \right).$$

52. Prove that

$$\left\{ \frac{1 + \sin \alpha + i \cos \alpha}{1 + \sin \alpha - i \cos \alpha} \right\}^n = \cos \left(\frac{n\pi}{2} - n\alpha \right) + i \sin \left(\frac{n\pi}{2} - n\alpha \right).$$

53. If $2 \cos \alpha = x + \frac{1}{x}$; $2 \cos \beta = y + \frac{1}{y}$, ...

prove that

$$xyz \ldots + \frac{1}{xyz \ldots} = 2 \cos (\alpha + \beta + \ldots).$$

54. If $2 \cos \frac{\pi}{2^r} = x_r + \frac{1}{x_r}$, show that

$$x_1 \cdot x_2 \cdot x_3 \cdots \text{ad inf} = -1.$$

55. Prove that

(i) $\sin^7 \theta = \frac{1}{64} \left[35 \sin\theta - 21 \sin 3\theta + 7 \sin 5\theta - \sin 7\theta \right]$.

(ii) $2^{11} \cos^5\theta \sin^7\theta = -\{ \sin 12\theta - 2 \sin 10\theta - 4 \sin 8\theta + 10 \sin 6\theta + 5 \sin 4\theta - 20 \sin 2\theta \}$.

(iii) $2^6 \cos^7\theta = \cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos\theta$.

56. Prove that $\sin^{-1} (\operatorname{cosec} \theta) = \left\{2n + (-1)^n\right\}\frac{\pi}{2} + i(-1)^n \log \cot\frac{\theta}{2}$

57. The voltage applied at the sending end of a long telephone wire being $V_0 \sin pt$ and the current entering the line is

$$V_0\sqrt{\frac{s + ikq}{r + ilq}} \sin qt.$$

If $r = 6$ ohms, $l = 3\times 10^{-3}$ henries, $k = 5 \times 10^{-9}$ farads. $s = 3\times 10^{-4}$ *mho*, $q = 6 \times 10^{8}$ find the current.

[Ans. $1.261 \times 10^{-3} V_0 \sin (6000t + 0.1111)$.]

58. The current C entering a telephone is given by

$$C = \frac{2V_0}{\left(R + \frac{r}{n}\right)e^{ln}}.$$

where $V_0 = 10 \sin 5000t$ and $n = \sqrt{rkqi}$.

If $r = 88$ ohms, $k = 5\times 10^{-8}$ farad, $q = 5000$, $l = 40$ and $R = 100 + 0.04qi$ find the values of a and b, assuming C has the form $a \sin (at + b)$.

[Ans. $a = 0.0002596$, $b = -3\cdot 798$]

59. If $\cos^{-1} (x+iy) = \alpha + i\beta$, prove that.

$x^2 \sec^2 \alpha - y^2 \operatorname{cosec}^2 \alpha = 1$ and $x^2 \operatorname{sech}^2 \beta + y^2 \operatorname{cosech}^2 \beta = 1$.

CHAPTER II

EQUATIONS OF THE FIRST ORDER AND OF THE FIRST DEGREE

2·1. We shall consider in this chapter, the simplest type of a differential equation, i.e. the ordinary differential equation of the first order and first degree. Such an equation can be written as

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M(x, y)\ dx + N(x, y)\ dy = 0$$

The various methods available to solve this equation can be classified as follows :

(i) Method of separation of variables.

(ii) Method for equations homogeneous in x and y.

(iii) Method for non-homogenous equations of the first degree in x and y.

(iv) Method for exact differential equations and those which can be made exact by the use of the integrating factors.

(v) Linear equations and those, which can be reduced to the linear form.

(vi) Method of substitution to reduce the equation to one of the above forms.

2·2. Separation of Variables : In the standard form of the equation above, if M is a function of x alone say $f_1(x)$ and N be a function of y alone say $f_2(y)$, then the equation is in the form

$$f_1(x)\ dx + f_2(y)\ dy = 0.$$

and its solution obtained by straight integration is

$$\int f_1(x)\ dx + \int f_2(y)\ dy = c.$$

Any equation of the form

$$f_1(x)\ \phi_2(y)\ dx + f_2(x)\ \phi_1(y)\ dy = 0.$$

can be expressed in the form "Variable separable" by dividing throughout by $f_2(x)\ \phi_2(y)$, viz.

$$\frac{f_1(x)}{f_2(x)}\ dx + \frac{\phi_1(y)}{\phi_2(y)}\ dy = 0$$

Example 1. $\dfrac{dy}{dx} = e^{x-2y}$.

Here we first separate out the variables.

The given equation is $\dfrac{dy}{dx} = e^{x} \cdot e^{-2y}$ $\quad \therefore \; e^{2y}\,dy - e^{x}\,dx = 0.$

Integrating $\frac{1}{2}e^{2y} - e^{x} = c$, which is the general solution of the differential equation.

Example 2. $3e^{x} \tan y \, dx + (1 - e^{x}) \sec^2 y \, dy = 0.$

Separating the variables by dividing throughout by $(1 - e^{x}) \tan y$ we have

$$\frac{3e^{x}}{1 - e^{x}}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Integrating, we have

$$-3 \log (1 - e^{x}) + \log \tan y = \log c$$

$\therefore \tan y = c\,(1 - e^{x})^{3}$ which is the solution of the equation

(Note how the arbitrary constant is introduced here for the sake of giving a neat form to the solution).

Examples : II—A

Solve : **Ans.**

1. $\left(y - x\dfrac{dy}{dx}\right) = a\left(y^2 + \dfrac{dy}{dx}\right)$. $\qquad (a + x)(1 - ay) = cy.$

2. $a\left(x\dfrac{dy}{dx} + 2y\right) = 2xy\dfrac{dy}{dx}$ $\qquad x^2 y = c e^{2y/a}$

3. $(x + 1)\dfrac{dy}{dx} + 1 = 2e^{-y}$ $\qquad (x + 1)e^{y} = 2x + c.$

4. $x \cos x \cos y + (\sin y)\dfrac{dy}{dx} = 0.$ $\qquad x \sin x + \cos x - \log(\cos y) = c.$

5. $x(1 - y)\,dx + (1 + y^2)(x - 1)\,dy = 0$

$$x + \log(x - 1) - \frac{y^2}{2} - y - \log(1 - y)^2 + c = 0.$$

6. $\dfrac{y}{x}\dfrac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2y^2}$ $\qquad \sqrt{1 + y^2} = \frac{1}{3}(1 + x^2)^{3/2} + c$

7. $\dfrac{dy}{dx} = \dfrac{x \sin x}{2e^{y} \sinh y}$ $\qquad e^{2y} - 2e^{y} = \sin x - x \cos x$

2·3. Equations homogeneous in x and y : These equations can be written in the form

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$$

where f_1 and f_2 are homogenous expressions in x and y and of the same degree, say r.

Then $f_1(x, y) = x^r f_1(y/x)$ and $f_2(x, y) = x^r f_2(y/x)$ and so the equation becomes

$$\frac{dy}{dx} = \frac{f_1(y/x)}{f_2(y/x)}$$

If we substitute therefore $\frac{y}{x} = v$ i. e. $y = vx$, the above equation becomes

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{f_2(v)}$$

The r. h. s. of this equation is only a function of v, say $F(v)$ so that

$$v + x\frac{dv}{dx} = F(v)$$

We can now separate the variables, leading to

$$\frac{dv}{F(v) - v} - \frac{dx}{x} = 0$$

and so integrate the equation. The solution will be in terms of v and x. By substituting y/x for v we get the required solution.

Example 1. $x^2y\,dx - (x^3 + y^3)\,dy = 0$.

The equation is homogeneous, so we write it as

$$\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}.$$

Putting $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it becomes

$$v + x\frac{dv}{dx} = \frac{v}{1+v^3}.$$

$$\therefore x\frac{dv}{dx} = \frac{v}{1+v^3} - v. \quad \text{or} \quad x\frac{dv}{dx} = \frac{-v^4}{1+v^3}.$$

$$\therefore \frac{1+v^3}{v^4}\,dv + \frac{dx}{x} = 0$$

$$\text{or} \left[\frac{1}{v^4} + \frac{1}{v}\right] dv + \frac{dx}{x} = 0$$

Integrating $-\frac{1}{3v^3} + \log v + \log x = \log c$

$$\therefore \log\left(\frac{vx}{c}\right) = \frac{1}{3v^3},$$

$$\therefore vx = c\,e^{\frac{1}{3v^3}}$$

Putting $v = y/x$, we have the solution of the differential equation

$$y = c\,e^{\frac{x^3}{3y^3}}.$$

Examples : II—B

Solve : (Ans. given on the right)

1. $(x^2 + y^2)\,dx - 2xy\,dy = 0$. $\qquad$ $x^2 - y^2 = cx$.
2. $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$. $\qquad$ $cy = e^{x/y}$.
3. $2xy\,dx + (y^2 - x^2)\,dy = 0$. $\qquad$ $x^2 + y^2 = cy$.
4. $(x + y)\dfrac{dy}{dx} + (x - y) = 0$. $\qquad$ $\log\sqrt{x^2 + y^2} + \tan^{-1}\dfrac{y}{x} = c$.
5. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$. $\qquad$ $cy = e^{y/x}$.
6. $y\sqrt{x^2+y^2}\,dx - x(x + \sqrt{x^2+y^2})\,dy = 0$.
   $cx - \sqrt{x^2 + y^2} = x\log(\sqrt{x^2 + y^2} - x)$.

2.4. Non-homogeneous linear equations : These can be written in the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}.$$

Using the transformation equations

$$x = X + h \text{ and } y = Y + k,$$

we have $\quad ax + by + c = aX + bY + (ah + bk + c)$

and $\quad a'x + b'y + c' = a'X + b'Y + (a'h + b'k + c')$

$$\left.\begin{aligned} ah + bk + c &= 0 \\ \text{and} \quad a'h + b'k + c' &= 0 \end{aligned}\right\} \text{ i. e. } h = \frac{bc' - b'c}{ab' - a'b},\; k = \frac{ca' - c'a}{ab' - a'b}$$

and provided $ab' - a'b \neq 0$, then the differential equation in terms of the new variables X and Y becomes

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

which is homogeneous and can be solved by the above article. If the solution of this equation be

$$f(X, Y) = c$$

then the solution of the original equation is

$$f(x-h,\ y-k)=c.$$

If $ab'-a'b=0$, then the values of h, k are infinite, and so the above method fails. Here $\frac{a}{a'}=\frac{b}{b'}$ which be put equal to say $\frac{1}{m}$,

$$\therefore\ \frac{a}{a'}=\frac{b}{b'}=\frac{1}{m}$$

$$\therefore\ a'=ma,\ b'=mb$$

and the differential equation in this case is

$$\frac{dy}{dx}=\frac{ax+by+c}{m(ax+by)+c'}.$$

Here we substitute $v=ax+by$ which gives

$$\frac{dv}{dx}=a+b\frac{dy}{dx}\ \text{or}\ \frac{dy}{dx}=\left(\frac{dv}{dx}-a\right)/b$$

$\therefore$ The differential equation becomes

$$\frac{\frac{dv}{dx}-a}{b}=\frac{v+c}{mv+c'}=\text{F}(v)\text{ say}$$

$$\therefore\ \frac{dv}{a+b\,\text{F}(v)}=dx$$

and so the equation can be integrated.

Example 1. $(3y-7x+7)\,dx+(7y-3x+3)\,dy=0.$

Putting $x=\text{X}+h$, and $y=\text{Y}+k$, the equation becomes

$$(3\text{Y}-7\text{X})\,d\text{X}+(7\text{Y}-3\text{X})\,d\text{Y}=0,$$

where h, k are given by

$$\left.\begin{array}{r} 3k-7h+7=0 \\ \text{and}\quad 7k-3h+3=0 \end{array}\right\}\quad \text{tha is}\quad \begin{array}{l} h=\dfrac{49-9}{-9+49}=1 \\ k=\dfrac{-21+21}{-9+49}=0 \end{array}$$

The equation being now homogeneous we put $\text{Y}=v\text{X}$ and the equation is

$$v+\text{X}\frac{dv}{d\text{X}}=\frac{7-3v}{7v-3}$$

$$\text{or}\quad \text{X}\frac{dv}{dx}=\frac{7(1-v^2)}{7v-3}$$

Separating the variables, $\frac{7v-3}{1-v^2}\,dv = 7\frac{dX}{X}$

$$\therefore \left[\frac{2}{1-v} - \frac{5}{1+v}\right] dv - 7\frac{dX}{X} = 0$$

Integrating $-2\log(1-v) - 5\log(1+v) - 7\log X = -\log c$

or $$(1-v)^2(1+v)^5X^7 = c$$

Putting $v = \frac{Y}{X}$ this becomes, $(X-Y)^2(X+Y)^5 = c$

and as $x = X+1$, and $y = Y$, the required solution is

$$(x-y-1)^6 = c.$$

Example 2. $(6x-4y+1)\frac{dy}{dx} = 3x-2y+1.$

$$\frac{dy}{dx} = \frac{3x-2y+1}{2(3x-2y)+1}$$

Here let $3x-2y = v$, $3-2\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \frac{3-\frac{dv}{dx}}{2} = \frac{v+1}{2v+1}$$

$$\therefore 3-\frac{dv}{dx} = \frac{2(v+1)}{2v+1}$$

$$\therefore \frac{dv}{dx} = \frac{4v+1}{2v+1} \text{ or } \frac{2v+1}{4v+1}\,dv = dx$$

$$\therefore \frac{1}{2}\left\{1+\frac{1}{4v+1}\right\}dv = dx$$

$$\therefore \frac{1}{2}\left\{v+\frac{1}{4}\log(4v+1)\right\} = x+c$$

$\therefore\ v+\frac{1}{4}\log 4+\frac{1}{4}\log(v+\frac{1}{4}) = 2x+c'$, where $c' = 2c$

or $v+\frac{1}{4}\log(v+\frac{1}{4}) = 2x+c''$ where c'' is a new constant

Substituting for v, in terms of x and y, we have the solution of the equation

$$3x-2y+\frac{1}{4}\log(3x-2y+\tfrac{1}{4}) = 2x+c''$$

or $$x-2y+\frac{1}{4}\log(3x-2y+\tfrac{1}{4}) = c''$$

Examples : II—C

Solve : **Ans.**

1. $(y-2x)\,dx+(2y-3x+1)\,dy = 0.$ $c(y^2-x^2-xy-4x+3y+1)^{\sqrt{5}} = \left[\frac{2y+4-(\sqrt{5}+1)(x+1)}{2y+4+(\sqrt{5}-1)(x+1)}\right]^2$

2. $\frac{dy}{dx}+\frac{2x+3y}{y+2} = 0$ $(2x+y-4)^2 = c(x+y-1).$

3. $(2x-y+1)\,dx+(2y-x-1)\,dy=0.$ $x^2-xy+y^2+x-y=c.$

4. $(6x+9y+6)\dfrac{dy}{dx}=2x+3y-1,\ (2x+3y+1)=c\,e^{(x-3y)}$

2·5. Exact differential Equation :— (a) *Definition* :— An exact differential equation of the first order is that equation which is obtained from its general solution by mere differentiation and without any additional process of elimination or reduction.

Thus $x^3y^4=c$ on differentiation gives

$$3x^2y^4\,dx+4x^3y^3\,dy=0$$

and in this form is called an exact differential equation. The other way in which we can understand an exact differential equation is that the differential expression is an exact differential of some function of x and y. Thus $3x^2y^4\,dx+4x^3y^3\,dy$ is an exact differential of x^3y^4. The above differential equation can be simplified to

$$3y\,dx+4x\,dy=0$$

but now it is not exact.

Thus the equation $M\,dx+N\,dy=0$ will be an exact differential equation if there be some function u of x and y, such that

$$Mdx+Ndy=du.$$

(b) *Condition of Exactness* :— We next want to see how to know whether a given equation is exact or not; for which sake we investigate the condition of exactness.

If $M\,dx+N\,dy=0$ be exact, then

$M\,dx+N\,dy=du$, where u is some function of x and y.

But $$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\therefore\ M=\frac{\partial u}{\partial x}\ \text{and}\ N=\frac{\partial u}{\partial y}$$

Elimination of u gives us

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\ \left(\text{as each}=\frac{\partial^2 u}{\partial x\partial y}\right)$$

This is the condition of exactness.

Thus in case of $3x^2y^4\,dx+4x^3y^3\,dy=0$, $M=3x^2y^4$, $N=4x^3y^3$

$$\frac{\partial M}{\partial y}=12x^2y^3=\frac{\partial N}{\partial x}$$

Hence the equation is exact.

But in case of $3y\,dx + 4x\,dy = 0$, $M = 3y$, $N = 4x$.

$$\frac{\partial M}{\partial y} = 3, \quad \frac{\partial N}{\partial x} = 4.$$

Thus here $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ and the equation is not exact.

(*c*) *Method of solution of an exact differential equation* :— The following rule can be given to solve an exact equation. Integrate $M\,dx$ as though y were constant, then integrate the terms in $N\,dy$ which do not contain x and equate the sum of these results to a constant.

The rule will now be proved, but those readers who are not interested in the proof can pass on to the examples

Let the solution of the equation $M\,dx + N\,dy = 0$ be $u = c$,

then
$$M = \frac{\partial u}{\partial x} \text{ and } N = \frac{\partial u}{\partial y}$$

Since $\dfrac{\partial u}{\partial x} = M$, integrating as though y were constant

it follows that

$$u = \int M\,dx + F(y) \quad \dots \quad \dots \quad \dots \quad (1)$$

where $F(y)$ is a function of y alone, to be determined.

Differentiating (1) with respect to y

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M\,dx + \frac{dF(y)}{dy}$$

But as $\dfrac{\partial u}{dy} = N$, $\therefore$ $\dfrac{dF(y)}{dy} = N - \dfrac{\partial}{\partial y}\displaystyle\int M\,dx$

and integrating with respect to y,

$$F(y) = \int\left\{N - \frac{\partial}{\partial y}\int M\,dx\right\}dy + A$$

Substituting this value of $F(y)$ in (1),

$$u = \int M\,dx + \int\left\{N - \frac{\partial}{\partial y}\int M\,dx\right\}dy + A$$

but $u = c$, therefore

$$\int M\,dx + \int \left\{ N - \frac{\partial}{\partial y} \int M\,dx \right\} dy = C$$

where $C = c - A$, is the solution.

Now $N - \dfrac{\partial}{\partial y} \displaystyle\int M\,dx$ cannot have x in any of its terms, for differentiating with respect to x gives $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ which must be zero since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

Therefore $\displaystyle\int \left\{ N - \frac{\partial}{\partial y} \int M dx \right\} dy$ is the same thing as $\displaystyle\int$ (Terms in N which do not contain x) dy. Therefore the rule is proved.

Example 1. $(2x^2+6xy-y^2)\,dx + (3x^2 - 2xy + y^2)\,dy = 0.$

Comparing the equation with the general form $M\,dx + N\,dy = 0$, it is seen that $M = 2x^2 + 6xy - y^2$ and $N = 3x^2 - 2xy + y^2$

Therefore $\dfrac{\partial M}{\partial y} = 6x - 2y$ and $\dfrac{\partial N}{\partial x} = 6x - 2y$

That is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ which shows that the equation is exact.

Apply the rule. Regarding y as constant and integrating M

$$\int M\,dx = \int (2x^2 + 6xy - y^2)\,dx = \frac{2}{3}x^3 + 3x^2y - y^2x$$

then integrating the terms in N dy which do not contain x

$$\int (\text{Terms in N which do not cantain } x)\,dy = \int y^2 dy = \frac{1}{3}y^3$$

and finally equating the sum of these integrals to a constant

$$\frac{2}{3}x^3 + 3x^2y - xy^2 + \frac{1}{3}y^3 = c$$

or $\qquad 2x^3 + 9x^2y - 3xy^2 + y^3 = c'$

is the required solution.

Differentiation will show that this solution is correct.

Examples : II—D

Solve :

1. $(a^2 - 2xy - y^2)\,dx - (x+y)^2 dy = 0.$
2. $(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})\,dy$
   $+ (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\,dx = 0.$

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3. $(x - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0.$

4. $(x^2 + y^2 - a^2)\, x\, dx + (x^2 - y^2 - b^2)\, y\, dy = 0.$

5. $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \dfrac{x}{y}\right) dy = 0.$

6. $(y^2 e^{xy^2} + 4x^3)\, dx + (2xye^{xy^2} - 3y^2)\, dy = 0$

7. $\dfrac{dy}{dx} = \dfrac{1 + y^2 + 3x^2y}{1 - 2xy - x^3}$

8. $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$

Ans.

1. $a^2x - \dfrac{y^3}{3} - xy^2 - x^2y = c.$ 2. $x^2y^2 + 4x^3y - 4xy^3 + y^3 - xe^y + e^{2x}y + x^4 = c.$

3. $3x^2 - 12x^2y - 12xy^2 + y^3 = c.$ 4. $x^4 - y^4 + 2x^2y^2 - 2a^2x^2 - 2b^2y^2 = c$

5. $x + ye^{x/y} = c.$ 6. $e^{xy^2} + x^4 - y^3 = c.$

7. $x^3y + xy^2 + x - y = c.$

8. $y \sin x + x \sin y + xy = c.$

2·6. Integrating factors :— If the equation $M dx + N dy = 0$ be not exact, it is possible to find a multiplying factor μ, which is a function of x and y such that the new equation formed is exact. Such a multiplying factor is called an integrating factor.

Thus consider the equation $(x - y)\, dx + x dy = 0.$

Here $M = x - y$, $N = x$, therefore $\dfrac{\partial M}{\partial y} = -1$ and $\dfrac{\partial N}{\partial x} = 1,$

therefore the equation is not exact.

If the equation is multiplied by $\dfrac{1}{x^2}$ it becomes

$$\left(\frac{1}{x} - \frac{y}{x^2}\right)dx + \frac{1}{x} = 0.$$

The new values of M and N are $M = \dfrac{1}{x} - \dfrac{y}{x^2}$ and $N = \dfrac{1}{x}$

form which $\dfrac{\partial M}{\partial y} = -\dfrac{1}{x^2}$ and $\dfrac{\partial N}{\partial x} = -\dfrac{1}{x^2}$, so $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

and the equation in the new form is exact and can be integrated by the method of article (2.5).

The multiplying factor $\dfrac{1}{x^2}$ is therefore the integrating factor of the given equation.

Example:— Given that the integrating factor of the equation

$$y\sec^2 x dx + \left[3\tan x - \left(\frac{\sec y}{y}\right)^2\right]dy = 0 \text{ is of the form } y^n.$$

Find n and hence solve the equation.
Multiplying by y^n the equation becomes

$$y^{n+1}\sec^2 x\, dx + y^n\left[3\tan x - \left(\frac{\sec y}{y}\right)^2\right]dy = 0.$$

Here $M = y^{n+1}\sec^2 x$, $N = y^n\left[3\tan x - \left(\frac{\sec y}{y}\right)^2\right]$

$$\therefore \frac{\partial M}{\partial y} = (n+1)y^n\sec^2 x, \quad \frac{\partial N}{\partial x} = 3y^n\sec^2 x.$$

For the equation to be exact $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\therefore$ $(n+1)y^n\sec^2 x = 3y^n\sec^2 x$ $\quad\therefore$ $(n+1) = 3$ or $n = 2$
$\therefore$ Integrating factor is y^2.
$\therefore$ The equation in the exact form is

$$y^3\sec^2 x\, dx + [3y^2\tan x - \sec^2 y]\, dy = 0$$

and the solution is

$$y^3\tan x - \tan y = c.$$

We have next to see how to find the I. F. of the given non exact equation of $Mdx + Ndy = 0$. The following rules will be found convenient to find the I. F.

(a) I. F. found by inspection :

In a number of problems, a little analysis helps to fix up the integrating factor, as illustrated in the following problems :-

Example 1. $(x^4 e^x - 2mxy^2)\, dx + 2mx^2 y dy = 0.$

It is easy to see that second term $-2mxy^2 dx$ must be associated with $2mx^2ydy$; and the I. F. should be such that it does not introduce 'y' in x^4e^x. So it is suggested that the I. F. is x^n, where n is to be fixed. By the method of the above example $n = -4$ and so the integrating factor is $\dfrac{1}{x^4}$.

Using this I. F. the example becomes

$$\left(e^x - 2m\frac{y^2}{x^3}\right)dx + 2m\frac{y}{x^2}dy = 0$$

or arranging the terms,

$$e^x dx - 2m\frac{y^2}{x^3}dx + 2m\frac{y}{x^2}dy = 0.$$

The last two terms originate from differentiation of the single term $a\frac{y^3}{x^3}$ and so integrating, the solution is

$$x^2 + a\frac{y^3}{x^3} = c.$$

Example 2. $y(2xy + e^x)\,dx - e^x\,dy = 0.$

Here it is easy to see that we should put $e^x y\,dx$ and $-e^x\,dy$ together, and term $2xy^2\,dx$ should not involve 'y' which suggests that $\frac{1}{y^2}$ may be an I. F.

Using this we have

$2x\,dx + \frac{e^x}{y}\,dx - \frac{e^x}{y^2}\,dy = 0$, and the equation is exact.

The last two terms arise from the single expression $\frac{e^x}{y}$ and so the required solution is

$$x^2 + \frac{e^x}{y} = c.$$

(b) I. F. for a homogeneous equation :

If $M\,dx + N\,dy = 0$ be a homogeneous equation then $\frac{1}{Mx+Ny}$ is the I. F. of the equation provided $Mx + Ny \neq 0$.

Example : $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

This equation is homogeneous, with $M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$.

$$\therefore \text{ I. F. is } \frac{1}{Mx + Ny} = \frac{1}{x(x^2y - 2xy^2) - y(x^3 - 3x^2y)}$$

$$= \frac{1}{x^2y^2}$$

Using this, the equation becomes

$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$ which can be seen to be exact ; so integrating it, the solution is

$$\frac{x}{y} - 2\log x + 3\log y = \log c$$

$$\therefore \log\frac{y^3}{cx^2} = -\frac{x}{y}$$

$$\text{or } y^3 = cx^2e^{-x/y}$$

Here it may be noted that there is the alternative method $y = vx$ for the homogeneous equation.

(c) I. F. for equation of the type $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$. If the equation $M dx + N dy = 0$ be of this type then $\dfrac{1}{Mx - Ny}$ is the I. F. provided $Mx - Ny \neq 0$.

Example. $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$.

Here $Mx - Ny = xy(xy + 2x^2y^2 - xy + x^2y^2)$
$= 3x^3y^3$.

$\therefore$ I. F. $\dfrac{1}{3x^3y^3}$.

Using this I. F. we have

$$\frac{1}{3}\left\{\frac{1}{x^2y^2}(xy + 2x^2y^2)\,dx + \frac{1}{x^3y^3}(xy - x^2y^2)\,dy\right\} = 0$$

$$\text{or } \left[\frac{1}{x^2y} + \frac{2}{x}\right]dx + \left[\frac{1}{xy^2} - \frac{1}{y}\right]dy = 0 \text{ which is exact}$$

and so integrating, the solution is

$$-\frac{1}{xy} + 2\log x - \log y = \log c$$

$$\text{or } x^2 = cye^{1/xy}.$$

(d) For the equation $M dx + N dy = 0$, if

$\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ be a function of x alone, say $f(x)$ then the I. F. of the equation is $e^{\int f(x)\,dx}$ and if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ be a function of y alone, say $F(y)$ then the I. F. of the equation is $e^{\int F(y)\,dy}$.

It may be noted here, that the possibility of such an I. F. if it exists, can be decided when finding out the exactness of the equation.

Example 1. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$.

Here $M = x^2 + y^2 + 2x \quad N = 2y$

$\therefore \dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{2y - 0}{2y} = 1$ and so can be considered as a function of x alone.

$\therefore$ The I. F. is $e^{\int 1 \cdot dx} = e^x$.

Using this I. F. the equation is $e^x(x^2 + y^2 + 2x)\,dx + 2e^x y\,dy = 0$ which being exact, integrating, we get

$$x^2e^x + e^x y^2 = c \text{ or } e^x(x^2 + y^2) = c.$$

Example 2. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0.$

Here $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = (6x^2y^3 - 2x) - (12x^2y^3 + 2x)$

$$= -6x^2y^3 - 4x = -2x(3xy^3 + 2)$$

$$\therefore \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{-2x(3xy^3 + 2)}{yx(3xy^3 + 2)} = -\frac{2}{y}$$ which is a function of y

alone.

$\therefore$ The I. F. is $e^{-\int \frac{2}{y}dy} = e^{-2\log y} = \dfrac{1}{y^2}$

Using this, the equation is

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

and so the solution is

$$x^3y^2 + \frac{x^2}{y} = c \text{ or } x^3y^3 + x^2 = cy.$$

(e) I. F. for the equation of the type

$$x^{\alpha}y^{\beta}\,(my\,dx + nx\,dy) + x^{\alpha'}y^{\beta'}\,(m'y\,dx + n'x\,dy) = 0.$$

If all the above four rules fail, and the example is or can be brought to this form, then an I. F. can be found which is of the form $x^{km-1-\alpha}y^{kn-1-\beta}$ (or $x^{k'm'-1-\alpha'}y^{k'n'-1-\beta'}$), where k (or k') has the value obtained from the simultaneous equations

$$\left.\begin{aligned} km - 1 - \alpha &= k'm' - 1 - \alpha' \\ kn - 1 - \beta &= k'n' - 1 - \beta' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2)$$

The explanation is that on multiplying the first term in the equation by $x^{km-1-\alpha}y^{kn-1-\beta}$ it becomes exact that is $\dfrac{1}{k}\,d(x^{km}y^{kn})$, similarly on multiplying the second term in the equation by $x^{km'-1-\alpha'}y^{k'n'-1-\beta'}$, it also becomes exact, which is $\dfrac{1}{k'}\,d(x^{k'm'}y^{k'n'})$, where k and k' may have any values. To have a common I. F. for the whole equation we put $x^{km-1-\alpha}y^{kn-1-\beta} = x^{k'm'-1-\alpha'}y^{k'n'-1-\beta'}$ that leads us

to the simultaneous equations (2) above, fixing the value of k (or k') Then $x^{km-1-\alpha}y^{kn-1-\beta}$ is the I. F. for the equation.

Example $(2x^2y^2+y)\,dx-(x^3y-3x)\,dy=0.$

It may be verified that the equation is not exact, and so recourse is to be made to find the I. F.

The equation being not homogeneous, or of the form $f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$, I. F. in (b) or (c) cannot be used; and as $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial M}{\partial x}}{N \text{ or } M}$ are not functions of x or y alone, we cannot use I. F. given in (d) above. We therefore try to arrange the example in the form

$$x^{\alpha}y^{\beta}(my\,dx+nx\,dy)+x^{\alpha'}y^{\beta'}(m'y\,dx+n'x\,dy)=0$$

and use $x^{km-1-\alpha}y^{kn-1-\beta}$ as the I. F.

It is natural to take out common factors from the two terms in the given equation and write it as

$$y(2x^2y+1)\,dx-x(x^2y-3)\,dy=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

which is written as $x^2y(2y\,dx-x\,dy)+(y\,dx+3x\,dy)=0$ and is in the required form.

Here $\alpha=2,\ \beta=1,\ m=2,\ n=-1,\ \alpha'=0,\ \beta'=0,\ m'=1,\ n'=3$

The I. F. for the first term in the equation is

$$x^{km-1-\alpha}y^{kn-1-\beta}=x^{2k-3}y^{-k-2}$$

and for the second term is

$$x^{k'm'-1-\alpha'}y^{k'n'-1-\beta'}=x^{k'-1}y^{3k'-1}.$$

To have these identical, we should have

$$\left.\begin{aligned} 2k-3&=k'-1\\ \text{and } -k-2&=3k'-1 \end{aligned}\right\}$$

Solving these we get $k=5/7$.

$\therefore$ I. F. for the given equation which is

$x^{2k-3}y^{-k-2}$ becomes $x^{-11/7}y^{-19/7}$

For the sake of integration, it is better to use this I. F. in the original form of the equation, than in the form (3) to which it is reduced. This makes the equation

$$(2x^{3/7}y^{-5/7}+x^{-11/7}y^{-12/7})\,dx-(x^{10/7}y^{-12/7}-3x^{-4/7}y^{-19/7})\,dy=0.$$

which is now exact. and so integrates into

$$\frac{x^{10/7}y^{-5/7}}{\text{[illegible]}}-\frac{x^{-4/7}y^{-12/7}}{4}=c.$$

Examples : II—E

Solve :

	Ans.
1. $y\,dx - x\,dy + \log x\,dx = 0.$	$cx + y + \log x + 1 = 0.$
2. $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0.$	$2\log x - \log y = \frac{1}{xy} + c$
3. $(x^2 + y^2)\,dx - 2xy\,dy = 0.$	$x^2 - y^2 = cx.$
4. $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$	$x^2y^2(y^2 - x^2) = c.$
5. $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0.$	$x^2 - y^2 - 1 = cx.$
6. $y(xy + 1)\,dx + x(1 + xy + x^2y^2)\,dy = 0.$	$2x^2y^2 \log y - 2xy - 1 = cx^2y^2$
7. $ye^y\,dx = (y^3 + 2xe^y)\,dy.$	$-\frac{x}{y^2} + e^y = c$

2·7. Linear equation and equations reducible to this form :— (*a*) *Linear Equation of the first order* :— When the dependent variable and its differential coefficients are in the first degree only and are not multiplied together, then a differential equation is said to be linear.

The general form of this equation is

$$\frac{dy}{dx} + Py = Q,$$

where P and Q are fuuctions of x or constants. It should be clear that the equation $R\frac{dy}{dx} + Sy = T$, where R, S and T are functions of x or constants, only requires to be divided through by R to bring to the general linear form.

First consider the case when $Q = 0$, then separating the variables

$$\frac{dy}{y} + P\,dx = 0.$$

Integrating

$$\log y + \int P\,dx = \log c$$ (log c being more convenient than c)

from which $\log \frac{y}{c} = -\int P\,dx$ or $\frac{y}{c} = e^{-\int P dx}$

$$\text{or } ye^{\int P dx} = c.$$

In what follows it is necessary to differentiate $e^{\int P dx}$. Let $e^{\int P dx} = e^u$, then $u = \int P\,dx$ and $\frac{du}{dx} = P$.

$$\therefore \frac{d}{dx} e^{\int P dx} = \frac{d}{dx} e^{u} = e^{u} \frac{du}{dx} = P e^{\int P dx}$$

Now differentiat $y e^{\int P dx} = c.$

then $$yP e^{\int P dx} + e^{\int P dx} \frac{dy}{dx} = 0$$

or $$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = 0.$$

Conversely the integral of the expression on the left side of this equation is $y\, e^{\int P dx}$. Therefore in the equation $\frac{dy}{dx} + Py = 0$ if $\frac{dy}{dx} + Py$ is multiplied by $e^{\int P dx}$ the integral of the resulting expression is $y\, e^{\int P dx}$; and $e^{\int P dx}$ is an integrating factor of the equation.

Returning to the general form and multipling each side of the equation by $e^{\int P dx}$,

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = e^{\int P dx} Q.$$

Integrating, $$y e^{\int P dx} = \int e^{\int P dx} Q dx + c.$$

Thus the solution of the linear differential equation is given by

$$\boxed{y \text{ (I.F.) } dx = c + \int Q \text{ (I.F.) } dx}$$

where $\text{I. F.} = e^{\int P dx}$

Example 1. $\frac{dy}{dx} - 2y = e^{2x}.$

Comparing the given equation with the general form $\frac{dy}{dx} + Py = Q$, it can be seen that $P = -2$ and therefore the I. F. is $e^{\int -2dx} = e^{-2x}$

Hence, we have

$$ye^{-2x} = c + \int e^{2x}\, e^{-2x} dx$$

i. e., $ye^{-2x} = c + x$; $\therefore y = e^{2x}(c + x)$.

Example 2. $x(1-x^2)\dfrac{dy}{dx} + (2x^2-1)y = x^3$.

Here we first divide by $x(1-x^2)$ through to bring equation to the linear form.

$$\therefore \frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}\, y = \frac{x^2}{1-x^2}.$$

and we see that $P = \dfrac{2x^2-1}{x(1-x^2)} = -\dfrac{1}{x} + \dfrac{1}{2(1-x)} - \dfrac{1}{2(1+x)}$.

$$\therefore \int P\,dx = -\log x - \tfrac{1}{2}\log(1-x) - \tfrac{1}{2}\log(1+x) = -\log x(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}$$

$$= -\log x(1-x^2)^{\frac{1}{2}}$$

$$\therefore \text{ I. F.} = e^{\int P dx} = e^{-\log x(1-x^2)^{\frac{1}{2}}} = \frac{1}{x(1-x^2)^{\frac{1}{2}}}.$$

Hence, we get

$$\frac{1}{x(1-x^2)^{\frac{1}{2}}}\, y = c + \int \frac{x}{(1-x^2)^{3/2}}\, dx$$

$$\text{or} \quad \frac{1}{x(1-x^2)^{\frac{1}{2}}}\, y = c + (1-x^2)^{-\frac{1}{2}}$$

$\therefore$ The solution is $y = cx(1-x^2)^{\frac{1}{2}} + x$.

(b) *Equation reducible to the linear form* :— (i) Equations of the form $\dfrac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x or constants, can be made linear by changing the dependen variable in the following way. This equation is known at Bernoulli's Equation.

Dividing by y^n, $\quad y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$.

Let $y^{1-n} = z$, then differentiating with respect to x,

$(1-n)\,y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$ from which $y^{-n}\dfrac{dy}{dx} = \dfrac{1}{1-n}\cdot\dfrac{dz}{dx}$

therefore $\dfrac{1}{1-n}\dfrac{dz}{dx} + Pz = Q$

or $$\frac{dz}{dx} + (1-n)\,Pz = (1-n)\,Q$$

which is a linear equation with z as the dependent variable.

Example 3. $\dfrac{dy}{dx} = x^3y^3 - xy.$

The equation is $\dfrac{dy}{dx} + xy = x^3y^3.$

or dividing by y^3, $y^{-3}\dfrac{dy}{dx} + xy^{-2} = x^3.$

Write $z = y^{-2}$. $\therefore \dfrac{dz}{dx} = -2y^{-3}\dfrac{dy}{dx}$ or $y^{-3}\dfrac{dy}{dx} = -\frac{1}{2}\dfrac{dz}{dx}.$

$\therefore$ The equation becomes $-\frac{1}{2}\dfrac{dz}{dx} + xz = x^3$

or $\dfrac{dz}{dx} - 2xz = -2x^3$ which is linear in z and x.

The I. F. is $e^{\int -2x\,dx} = e^{-x^2}.$ Hence the solution is

$$ze^{-x^2} = c - 2\int e^{-x^2}x^3\,dx.$$

[The integral $-2\int e^{-x^2}x^3\,dx$ can by evaluated be substituting $-x^2 = t$, $-2x\,dx = dt$, and the integral is $e^{-x^2}(x^2+1)$]

$$\therefore\ z = ce^{x^2} + (x^2+1).$$

But $z = y^{-2}$, therefore the required solution is

$$\frac{1}{y^2} = ce^{x^2} + (x^2+1)$$

(*ii*) Another type of equation which can be reduced to the linear form is

$$f'(y)\frac{dy}{dx} + Pf(y) = Q,$$

where P and Q are functions of x alone or constants, and

$$f'(y) = \frac{d}{dy}f(y)$$

If we substitute $z = f(y)$, then $\dfrac{dz}{dx} = f'(y)\dfrac{dy}{dx}$ and so the

equatian becomes

$$\frac{dz}{dx} + Pz = Q,$$

which is linear.

Example. $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^2 x.$

Dividing by $\cos y$.

$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x.$$

Writing $z = \sec y$, so that $\dfrac{dz}{dx} = \sec y . \tan y \dfrac{dy}{dx}$

the equation is

$$\frac{dz}{dx} + z \tan x = \cos^2 x$$

Which is linear, with the I. F. $e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x.$

$\therefore$ The solution of the equation is $z \sec x = c + \int \cos^2 x \sec x \, dx.$

$$z \sec x = c + \sin x$$

or $\quad \sec y \sec x = c + \sin x$

or $\quad \sec y = (c + \sin x) \cos x$ is the solution.

Examples : II—F

Solve : Ans.

1. $x \dfrac{dy}{dx} + 3y = x^4 e^{1/x} y^3.$ $\qquad y^2 x^6 (e^{1/x^2} + c) = 1.$

2. $\dfrac{dy}{dx} + x^2 y = x^5.$ $\qquad e^{x^3/3}(y - x^3 + 3) = c.$

3. $3y^2 \dfrac{dy}{dx} + 2xy^3 = 4x e^{-x^2}$ $\qquad y^3 e^{x^2} = 2x^2 + c.$

4. $(1+x^2)\, dy = (\tan^{-1} x - y)\, dx.$ $\qquad y = \tan^{-1} x - 1 + c e^{-\tan^{-1} x}.$

5. $3x(1-x^2) y^2 \dfrac{dy}{dx} + (2x^2 - 1) y^3 = ax^3.$ $\qquad y^3 = ax + cx\sqrt{1-x^2}$

6. $x dy - \left\{ y + xy^3 (1 + \log x) \right\} dx = 0.$ $\qquad \dfrac{x^2}{y^2} = -\dfrac{2}{3} x^3 \left(\dfrac{2}{3} + \log x \right) + c$

7. $\dfrac{dy}{dx} \cos x + y \sin x = (y \sec x)^{\frac{1}{2}}$ $\qquad 4y = (c + \tan x)^2 \cos x$

8. $e^{-y} \sec^2 y \, dy = dx - x dy$ $\qquad x e^{y} = c + \tan y$

9. $(1-x^2) \dfrac{dy}{dx} = xy(1 + x^2 y^2).$ $\qquad y^2 + 1 = (1-x^2) y^2 [c - \log y (1-x^2)]$

10. $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$ $\qquad y = \frac{(c + x^2 - x)x^2}{x-1}$

11. $x^2 \frac{dy}{dx} = 3x^2 - 2xy + 1$ $\qquad y = \frac{c}{x^2} + x + \frac{1}{x}$

2.8 Method of substitution :— It may be convenient here to see how a given equation, $M dx + N dy = 0$ may be tackled, in general.

When the equation is to be solved, it is first necessary to classify it properly, and the following order may be tried.

(i) Separation of variables.

(ii) Exact, here one may try the possibility of the I.F. involving

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M \text{ or } N}$$

(iii) Homogeneous ($y = vx$ or I. F.).

(iv) Non-homogeneous Linear

(v) Linear or reducible to linear form.

(vi) and possibility of the remaining to I. F.

If the example does not come in any of the above forms the equation requires some substitution of variables, which will bring it to one of the above forms. Though no hard and fast rules can be given about the methed of substitutien, the following few hints may be found generally useful.

(*a*) If the equation be of the type $\frac{dy}{dx} = f(ax + by + c)$ a substitution $u = ax + by + c$ will be useful.

Example. $(x - y)^2 \frac{dy}{dx} = a^2$.

Here $u = x - y$, gives $\frac{du}{dx} = 1 - \frac{dy}{dx}$

$$\therefore u^2\left(1 - \frac{du}{dx}\right) = a^2.$$

This reduces to $\frac{u^2 du}{u^2 - a^2} = dx$.

$$\text{or } \left(1 + \frac{a^2}{u^2 - a^2}\right) du = dx$$

which on integration gives

$$u + \frac{a}{2} \log \frac{u-a}{u+a} = x + c$$

$$\text{or} \quad x - y + \frac{a}{2} \log \frac{x-y-a}{x-y+a} = x + c$$

leading to required solution

$$\frac{a}{2} \log \frac{x-y-a}{x-y+a} = y + c.$$

(*b*) If the differential equation involves as a factor $(xdy - ydx)$ then the substitution $v = \frac{y}{x}$ or $\frac{x}{y}$ will be useful; if it involves $(xdx + ydy)$ then $u = x^2 + y^2$ may be found a useful substitution. If the equation involves both the factors $(xdx + ydy)$ and $(xdy - ydx)$, then the polar transformation $x = r \cos \theta, y = r \sin\theta$ will be useful. It may be noted here that

$$xdx + ydy = rdr$$

$$\text{and} \quad xdy - ydx = r^2 d\theta$$

Example. $xdx + ydy = m\,(xdy - ydx)$.

Here putting $x = r \cos \theta, y = r \sin \theta$, the equation becomes

$$rdr = m\, r^2 d\theta \quad \text{or} \quad \frac{dr}{r} = m\, d\theta.$$

and the integral is $\log r = m\,\theta + c$

$$\text{or} \quad \log(x^2 + y^2) = 2m \tan^{-1}\frac{y}{x} + c'.$$

(*c*) Sometimes interchange of the dependent and independent variables, puts the equation in the standard form, particularly the Bernoulli's form.

Example. $(x^2y^3 + xy)\, dy = dx.$

We generally take y as dependent variable, then this equation cannot be classified into any of the standard forms. We, therefore, take y as an independent variable and write the equation as

$$\frac{dx}{dy} = x^2y^3 + xy$$

$$\text{or} \quad \frac{dx}{dy} - yx = x^2y^3$$

which is a form reducible to linear equation.

Dividing by x^2, $\quad x^{-2}\frac{dx}{dy} - x^{-1}y = y^3.$

Put $\quad z = x^{-1}$, so that $\frac{dz}{dy} = -x^{-2}\frac{dx}{dy}$

the equation becomes

$$\frac{dz}{dy} + yz = -y^3$$

$$\therefore \text{ I. F. is } e^{\int y\,dy} = e^{\frac{1}{2}y^2}$$

$$\therefore \text{ The integral is } z.\,e^{\frac{1}{2}y^2} = c - \int e^{\frac{1}{2}y^2}\,y^3 dy.$$

The last integral, which we have come across, can be evaluated by the substitution $\frac{1}{2}y^2 = t$, and so we have

$$ze^{\frac{1}{2}y^2} = c - e^{\frac{1}{2}y^2}(y^2 - 2)$$

$$\text{or} \qquad z = ce^{-\frac{1}{2}y^2} - y^2 + 2.$$

Since $z = x^{-1}$, so the required solution is

$$x^{-1} = ce^{-\frac{1}{2}y^2} - y^2 + 2.$$

(*d*) Sometimes a substituion is suggested by the form of the equation. An example will make this point clear.

Example 1. $(2 + 2x^2y^{\frac{1}{2}})\ y\,dx + (x^2y^{\frac{1}{2}} + 2)\,x\,dy = 0$

Here the term $x^2y^{\frac{1}{2}}$ occurs prominently in the equation, and the substitution $x^2y^{\frac{1}{2}} = v$ is suggisted.

Putting $x^2y^{1/2} = v$ or $y = v^2/x^4$

So that $dy = \dfrac{2v}{x^4}\,dv - \dfrac{4v^2}{x^5}\,dx$, and reduced the equation to

$$(2 + 2v)\frac{v^2}{x^4}\,dx + x\,(v + 2)\left(\frac{2v}{x^4}\,dv - \frac{4v^2}{x^5}\,dx\right) = 0$$

or $v(3 + v)\,dx - x\,(v + 2)\,dv = 0$

$$\text{Then } \frac{dx}{x} - \frac{2}{3}\frac{dv}{v} - \frac{1}{3}\frac{dv}{v+3} = 0$$

$\therefore$ $3 \log x - 2 \log v - \log (v + 3) = \log c$

$\therefore$ $x^3 = cv^2\,(v + 3)$

or $x^3 = cyx^4\,[x^2y^{\frac{1}{2}} + 3]$

or $xy\,(x^2y^{\frac{1}{2}} + 3) = c'$, where $c' = \dfrac{1}{c}$

Example 2. $(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0.$

The suggested transformation here is $x^2 = u,\ y^2 = v$ and this reduces the equation to

$$(2u + 3v - 7)\,du - (3u + 2v - 8)\,dv = 0$$

which is non-homogeneous linear.

The further transformation $u = s + 2,\ v = t + 1$ yields the homogeneous equation $(2s + 3t)\,ds - (3s + 2t)\,dt = 0$, and the transformation $s = rt,\ ds = r\,dt + t\,dr$ yields

$$2\,(r^2 - 1)\,dt + (2r + 3)\,t\,dr = 0.$$

Separating the variables, we have $2\,\dfrac{dt}{t} + \dfrac{2r+3}{r^2-1}\,dr = 0$

or $$2\,\frac{dt}{t} - \frac{1}{2}\,\frac{dr}{r+1} + \frac{5}{2}\,\frac{dr}{(r-1)} = 0$$

Then $4 \log t - \log(r + 1) + 5 \log(r - 1) = \log c$

$$\therefore\quad \frac{t^4\,(r-1)^5}{r+1} = \frac{(s-t)^5}{s+t} = \frac{(u-v-1)^5}{u+v-3} = \frac{(x^2-y^2-1)^5}{x^2+y^2-3} = c$$

and so the solution is $$(x^2 - y^2 - 1)^5 = c\,(x^2 + y^2 - 3).$$

Examples : II—G

Solve : **Ans.**

1. $\left(\dfrac{x+y-a}{x+y+b}\right)\dfrac{dy}{dx} = \dfrac{x+y+a}{x+y-b}$. $\qquad (x+y)^2 + ab = ce^{2(y-x)/(a+b)}$

2. $\dfrac{dy}{dx} + x\tan(y-x) = 1$. $\qquad e^{x^2/2}\sin(y-x) = c$.

3. $\dfrac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^2$. $\qquad c(y-x)e^{x^2/2} = (y-x)(x^2+2) + 1$.

4. $\dfrac{dy}{dx} = 1 + \dfrac{y}{x} - \cos\dfrac{y}{x}$. $\qquad \cot(y/2x) + \log x = c$.

5. $\left(2x\sinh\dfrac{y}{x} + 3y\cosh\dfrac{y}{x}\right)dx - 3x\cosh\dfrac{y}{x}\,dy = 0$. $\qquad x^2 = c\sinh^3\dfrac{y}{x}$

6. $x^2(x\,dx + y\,dy) + y(x\,dy - y\,dx) = 0$. $\qquad (x^2+y^2)(x+1)^2 = cx^2$.

7. $\dfrac{dy}{dx} = \dfrac{y+1}{(y+2)e^y - x}$. $\qquad (y+1)(e^y - x) = c$.

8. $x^4\dfrac{dy}{dx} + x^3y = -\sec(xy)$. $\qquad \sin(xy) = \dfrac{1}{2x^2} + c$.

9. $x\,dy - y\,dx = (x^2+y^2)(x\,dx + y\,dy)$. $\qquad \tan^{-1}\dfrac{y}{x} = \dfrac{1}{2}(x^2+y^2) + c$

Examples : II—H

Solve : **Ans.**

1. $\dfrac{dy}{dx} - y\tan x = y^4\sec x$. $\qquad \left[\dfrac{1}{y^3} = c\cos^3 x - \sin x\,(2\cos^2 x + 1)\right]$

2. $\dfrac{dy}{dx} = -\dfrac{4x^3y^2 + y\cos(xy)}{2x^4y + x\cos(xy)}$ $\qquad [\,x^4y^2 + \sin(xy) = c\,]$

3. $\dfrac{dy}{dx} + 2xy = 2e^{-x^2}$ $\quad [ye^{x^2} = c + 2x]$

4. $\cos x \dfrac{dy}{dx} + y + \sin x = 1.$ $\quad [(\sec x + \tan x)\, y = x + c]$

5. $x^3 \dfrac{dy}{dx} + 3y^2 = xy^2.$ $\quad \left[y - x = \dfrac{3y}{2x} + cxy\right]$

6. $y - x\dfrac{dy}{dx} = 3\left[1 + x^2\dfrac{dy}{dx}\right]$ $\quad [(y-3)(3x+1) = cx]$

7. $(x^2y+y^4)\,dx + (2x^3+4y^3x)\,dy = 0.$ $\quad [x^{7/2}y^{11}(7x^2 + 11y^3) = c]$

8. $(1+xy^2)\,dx + (1+x^2y)\,dy = 0.$ $\quad [x^2y^2 + 2(x+y) = c]$

9. $y^2(x^2+2)dx + (x^3+y^3)(ydx - xdy) = 0.$

$[2x^2y \log x - 2y + 2x^3 - y^3 = cx^2y]$

10. $xy - \dfrac{dy}{dx} = y^3e^{-x^2}.$ $\quad [y^{-2} = (2x + c)\,e^{-x^2}]$

11. $\dfrac{2y}{x}\,dx + (2\log x - y)\,dy = 0.$ $\quad [4y \log x - y^2 = c]$

12. $\sec x \dfrac{dy}{dx} = y + \sin x.$ $\quad [y = ce^{\sin x} - \sin x - 1]$

13. $\dfrac{dy}{dx} = \dfrac{y^3}{e^{2x} + y^2}.$ $\quad [y^2 = e^{2x}(c - 2\log y)]$

14. $\dfrac{2dy}{dx} + \cos^2(x - 2y) = 1.$ $\quad [\tan(x - 2y) = x + c]$

15. $\cos y - x \sin y \dfrac{dy}{dx} = \sec^2 x.$ $\quad [x \cos y = c + \tan x]$

16. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$ $\quad \left[xy + \dfrac{2x}{y^2} + y^2 = c\right]$

17. $\sin x \dfrac{dy}{dx} + 2y = \tan^3(x/2).$ $\quad \left[y = c\cot^2(x/2) + \dfrac{1}{5}\tan^3(x/2)\right]$

18. $\left\{y\left(1 + \dfrac{1}{x}\right) + \cos y\right\}dx + \left\{x + \log x - x\sin y\right\}dy = 0.$

$[xy + y\log x + x\cos y = c]$

19. $(x+y)^2\left(x\dfrac{dy}{dx} + y\right) = xy\left(1 + \dfrac{dy}{dx}\right).$ [Substitute $x + y = u$, $xy = v$

$\log xy = c - \dfrac{1}{x+y}$]

20. $m\left(x - y\dfrac{dy}{dx}\right) + n\left(y - x\dfrac{dy}{dx}\right) = 0.$ $\quad [(x+y)^{n-m} = c(x-y)^{m+n}]$

21. $x^2\dfrac{dy}{dx} + xy + \sqrt{1 - x^2y^2} = 0$ $\quad [\sin^{-1}(xy) + \log x = c]$

22. $y(x^2+y^2+a^2)\dfrac{dy}{dx}+x(x^2+y^2-a^2)=0.$ $[x^4+2x^2y^2-2a^2x^2+2a^2y^2+y^4=c]$

23. $xy^3\,dy=(xe^x-\frac{1}{4}y^4)\,dx.$ $[xe^x-e^x-\frac{1}{4}xy^4=c]$

24. $(x+2y^3)\dfrac{dy}{dx}=y.$ $[x=y^3+cy]$

25. $(x-y^2)+2xy\dfrac{dy}{dx}=0.$ $[y^2-cx=-x\log x]$

26. $\dfrac{dy}{dx}=e^{x-y}(e^x-e^y).$ $[e^y=e^x-1+ce^{-e^x}]$

27. $\dfrac{x\,dx+y\,dy}{x\,dy-y\,dx}=\sqrt{\left\{\dfrac{a^2-x^2-y^2}{x^2+y^2}\right\}}$

$\left[\sqrt{(x^2+y^2)}=a\sin\left(\tan^{-1}\dfrac{y}{x}+c\right)\right]$

28. $x^3\dfrac{dy}{dx}=y^3+y^2\sqrt{y^2-x^2}.$ $[y+\sqrt{(y^2-x^2)}=cxy]$

29. $(4x+y)^2\dfrac{dx}{dy}=1.$ $\left[c+4x=2\tan^{-1}\dfrac{4x+y}{2}\right]$

30. $r\sin\theta-\dfrac{dr}{d\theta}\cos\theta=r^2.$ $\left[\dfrac{1}{r}=c\cos\theta+\sin\theta\right]$

31. $(x\sqrt{x^2+y^2}-y)\,dx+(y\sqrt{x^2+y^2}-x)\,dy=0.$

$[(x^2+y^2)^{3/2}-3xy=c]$

32. $(xy^2-e^{1/x^3})\,dx-x^2y\,dy=0.$ $[2x^2e^{1/x^3}-3y^2=cx^2]$

33. $y(1+2xy)\,dx+x(1-xy)\,dy=0.$ $[y=cx^2e^{-1/xy}]$

34. $(x-2\sin y+3)\,dx+(2x-4\sin y-3)\cos y\,dy=0.$

$[8\sin y+4x+9\log(4x-8\sin y+3)=c]$

35. $(2xy^4e^y+2xy^3+y)\,dx+(x^2y^4e^y-x^2y^2-3x)\,dy=0.$

$\left[x^2e^y+\dfrac{x^2}{y}+\dfrac{x}{y^3}=c\right]$

36. $x\,dx+y\,dy+4y^3(x^2+y^2)\,dy=0.$ $\left[\text{I. F.}=\dfrac{1}{x^2+y^2},\ (x^2+y^2)e^{2y^4}=c\right]$

37. $\sin y\dfrac{dy}{dx}=\cos x(2\cos y-\sin^2x).$

$[\cos y=\frac{1}{2}\sin^2x-\frac{1}{2}\sin x+\frac{1}{4}+ce^{-2\sin x}]$

38. $x\dfrac{dy}{dx}-y+3x^3y-x^2=0.$ $\left[y=xe^{-x^3}\int e^{x^3}dx+cxe^{-x^3}\right]$

39. $(y+e^y-e^{-x})\,dx+(1+e^y)\,dy=0.$ $[y+e^y=(x+c)e^{-x}]$

40. $y\,dx+x(1-3x^2y^2)\,dy=0.$ $[y^6=ce^{-1/(x^2y^2)}]$

41. $y\log y\,dx+(x-\log y)\,dy=0.$ $[2x\log y=(\log y)^2+c]$

42. $\sin y \dfrac{dy}{dx} = (1 - x\cos y)\cos y.$ [$\sec y = x + 1 + ce^x$]

43. $(1+\sin y)\dfrac{dx}{dy} = [2y\cos y - x(\sec y + \tan y)].$ [$(\sec y + \tan y)x = y^2 + c$]

44. $4x^2 y \dfrac{dy}{dx} = 3x(3y^2+2) + (3y^2+2)^3.$ [$4x^6 = (c-3x^4)(3y^2+2)^2$]

45. $\dfrac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1.$ [$(x+y)^{-2} = x^2 + 1 + ce^{x^2}$]

46. $\dfrac{dz}{dx} + \dfrac{z}{x}\log z = \dfrac{z}{x^2}(\log z)^2$ $\left[\log z = \dfrac{2x}{Cx^2+1}\right]$

[Hint : Put $y = \log z$).

47. $(x+y-2)\,dx + (x-y+4)\,dy = 0.$

[Ans. $x^2 + 2xy - 4x - y^2 + 8y = c^2$]

48. $(ye^{xy} - \cos x)\,dx + xe^{xy}\,dy = 0.$ [Ans. $e^{xy} + \sin x = c$]

49. $e^{y-x} = \dfrac{dy}{dx} - xe^{(x+y)}$ [Ans. $e^{-y} = c + e^{-x} - (x-1)e^x$]

50. Find the constant n such that $(x+y)^n$ is an integrating factor of $(4x^2 + 2xy + 6y)\,dx + (2x^2 + 9y + 3x)\,dy = 0$ and hence solve the equation

[$n = 1$, $x^4 + 2x^3y + x^2y^2 + 3x^2y + 6xy^2 + 3y^3 = c$]

51. If y^n is an integrating factor of the equati

$$y(2x^2y + e^x)\,dx - (e^x + y^3)\,dy = 0$$

find n and hence solve the equation.

$\left[\text{Ans. } n = -2,\ \dfrac{2x^3}{3} + \dfrac{e^x}{y} - \dfrac{y^2}{2} = c\right]$

52. $x\dfrac{dy}{dx} = y + x\cos^2\dfrac{y}{x}$ [$y = x\tan^{-1}(\log cx)$]

53. $\left(\dfrac{y^2}{1+x^2} - 2y\right)dx + (2y\tan^{-1}x - 2x + \sinh y)\,dy = 0$

[$y^2\tan^{-1}x - 2xy + \cosh y = c$]

54. $r\dfrac{d\theta}{dr} + \tan\theta = \left(1 - r\tan\theta\dfrac{d\theta}{dr}\right) r\sin\theta$ [$r\sin\theta = Ae^{r\cos\theta}$]

55. $\dfrac{dy}{dx} + \dfrac{x(x^2+y^2-1)}{y(x^2+y^2+1)} = 0$ [Ans. $(x^2+y^2)^2 - 2(x^2-y^2) = c.$]

56. $xy(1+xy^2)\dfrac{dy}{dx} = 1$ [Ans. $x(ce^{-y^2/2} - y^2 + 2) = 1$]

57. $xy^2\dfrac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$ [Ans. $2\log x - x^2 - y^2 + \log(1+y^2) = c$]

CHAPTER IV

DIFFRENTIAL EQUATIONS OF THE FIRST ORDER AND OF A DEGREE HIGHER THAN THE FIRST

4. 1. Introduction :— We shall consiuer in this chapter the differential equations of the first order which are of a degree higher than the first. Using $p = \dfrac{dy}{dx}$, such an equation can be denoted by

$$f(x, y, p) = 0$$

or if the equation be algebraic, and of degree n, as

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots . P_n = 0 \qquad \ldots (1)$$

where $P_1, P_2, \ldots\ldots P_n$ are functions of x and y.

[It may be noted here that $p^2 \neq \dfrac{d^2y}{dx^2}$ but $\left(\dfrac{dy}{dx}\right)^2$ etc.]

The methods for solving such an equation can be broadly classified as

(a) Solvable for p
(b) Solvable for y
and (c) Solvable for x.

After developing these methods, we shall then consider the other methods available for solving such an equation.

4. 2. Solvable for p (or the method of factors) :— The differential equation (1) being of the nth degree in p, may be factorised into n linear factors, say

$$(p - F_1)\ (p - F_2)\ (p - F_3) \ldots\ldots (p - F_n) = 0 \quad \ldots (2)$$

where $F_1, F_2, \ldots . F_n$ are functions of x and y.

The equation (2) is equivalent to

$$(p - F_1) = 0,\ (p - F_2) = 0, \ldots\ldots (p - F_n) = 0 \quad \ldots (3)$$

each of which is a differential equation of the first order and the first degree and so can be integrated by the methods

of Chapter II. If the individual solutions of the equations (3) are

$$f_1(x, y, c) = 0, f_2(x, y, c) = 0, \ldots . f_n(x, y, c) = 0,$$

then the solution of the original differential equation is given by the product of these solutions and is

$$f_1(x, y, c) . f_2(z, y, c) \ldots \ldots f_n(x, y, c) = 0.$$

Example 1. Solve $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$.

Solving for p, we get

$$p = \frac{-(3x^2 - 2y^2) \pm \sqrt{(3x^2 - 2y^2)^2 + 24x^2y^2}}{2xy}$$

$$= \frac{-(3x^2 - 2y^2) \pm (3x^2 + 2y^2)}{2xy}$$

$$= \frac{2y}{x} \text{ or } \frac{-3x}{y}$$

Now $p = \dfrac{2y}{x}$ gives $\dfrac{dy}{y} - \dfrac{2dx}{x} = 0$ or $\log y - 2 \log x = \log c$

$\therefore \dfrac{y}{x^2} = c$ or $y - cx^2 = 0$ .. .. (i)

and $p = \dfrac{-3x}{y}$ gives $ydy + 3\,xdx = 0$

$\therefore \dfrac{y^2}{2} + \dfrac{3x^2}{2} = c'$ or $y^2 + 3x^2 - c = 0$ .. .. .. (ii)

From (i) and (ii), the required solution of the given differential equation is

$$(y - cx^2)\ (y^2 + 3x^2 - c) = 0.$$

Example 2. solve $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$.

The equation on simplification becomes

$$\frac{p^2 - 1}{p} = \frac{x^2 - y^2}{xy} \quad \text{or} \quad xyp^2 - (x^2 - y^2)p - xy = 0.$$

On factorizing $(xp + y)(yp - x) = 0$.

Taking $xp + y = 0$, we have $\dfrac{dy}{dx} = \dfrac{-y}{x}$ or $\dfrac{dy}{y} + \dfrac{dx}{x} = 0$

which gives on integration, $xy - c = 0$ .. .. (i)

The other factor $yp - x = 0$, gives $\dfrac{dy}{dx} = \dfrac{x}{y}$ or $xdx - ydy = 0$

which gives on integration, $x^2 - y^2 - c = 0$ .. .. (ii)

From (i) and (ii) the solution of the given diffential equation is

$$(xy - c)\ (x^2 - y^2 - c) = 0.$$

Example 3. Solve $p^3 - (x^2 + xy + y^2)\, p^2 + (x^3y + x^2y^2 + xy^3)\, p - x^3y^3 = 0$

The given equation is

$$p^3 - (x^2 + xy + y^2)\, p^2 + xy\,(x^2 + xy + y^2)\, p - x^3y^3 = 0 \quad \dots \quad \dots \quad \dots \quad \text{(i)}$$

For convenience of factorisation, let $a = x^2 + xy + y^2$ and $b = xy$, so that the equation (i) is

$$p^3 - ap^2 + abp - b^3 = 0.$$

and taking the first and the last term together, we have

$$(p - b)\,[p^2 + p\,(b - a) + b^2] = 0$$

$$\therefore\ p = b \text{ or } \frac{a - b \pm \sqrt{a^2 - 2ab - 3b^2}}{2}$$

Using the values of a and b, we have

$$p = xy, \text{ or } x^2 \text{ or } y^2$$

$$p = xy \text{ gives } \frac{dy}{y} = x\,dx \text{ or } \log y = \frac{x^2}{2} + \log c$$

$$\text{or } y - c e^{\frac{x^2}{2}} = 0 \quad \dots \quad \dots \quad \text{(ii)}$$

$$p = x^2 \text{ gives } dy - x^2 dx = 0 \text{ i.e. } 3y - x^3 - c = 0 \quad \dots \quad \dots \quad \text{(iii)}$$

$$\text{and } p = y^2 \text{ gives } \frac{dy}{y^2} - dx = 0 \text{ i. e. } -\frac{1}{y} - x = c$$

$$\text{or } xy + 1 + cy = 0 \quad \dots \quad \dots \quad \text{(iv)}$$

From (ii), (iii), (iv) the solution of the equation is

$$\left(y - c e^{\frac{x^2}{2}}\right)(3y - x^3 - c)\,(cy + xy + 1) = 0.$$

Examples : IV — A

Solve **Ans.**

1. $3p^2y^2 - 2xyp + 4y^2 - x^2 = 0.$ $[3(x^2 + y^2) \pm 4cx + c^2 = 0.]$

2. $p^2 + 2py \cot x = y^2.$ $\left[\left(y \sin^2 \frac{x}{2} - c\right)\left(y \cos^2 \frac{x}{2} - c\right) = 0\right]$

3. $p\,(p + y) = x\,(x + y)$ $[(y - \frac{1}{2}x^2 + c)\,(y + x + ce^{-x} - 1) = 0]$

4.3. Solvable for y :— If it is inconvenient to solve the equation for p, it may be convenient to solve it for y, that is y is expressible as a function of x and p, say

$$y = F\,(x, p) \quad \dots \quad \dots \quad \dots \quad \dots \quad (4)$$

Differentiating (4) w. r. t. x, we have

$$p = \frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} \cdot \frac{dp}{dx} \quad \dots \quad \dots \quad \dots \quad (5)$$

Equation (5) is a differential equation of first order in p and x and may be integrated, to say

$$f\,(x, p, c) = 0. \quad \dots \quad \dots \quad \dots \quad (6)$$

The elimination of p from the original equation (4) and the equation (6)gives the relation between the variables, independent of the derivative and so constitutes the solution of the differential equation (6). Sometimes the elimination of p from equations (4) and (6) is not convenient, then the solution is given by (4) and (6), considering p as a parameter.

Example 1. Solve $16x^2 + 2p^2y - p^3x = 0$.

Here it is not convenient to solve for p, and so we solve it for y, Solving the equation for y, we have,

$$2y = px - 16\frac{x^2}{p^2} \quad \ldots \quad \text{(i)}$$

Differentiating (i) w. r. t. x,

$$2p = p + x\frac{dp}{dx} - \frac{32x}{p^2} + \frac{32x^2}{p^3}\frac{dp}{dx}$$

This simplifies to

$$p\,(p^2 + 32x) - x\,(p^2 + 32x)\frac{dp}{dx} = 0$$

$$\text{or } (p^2 + 32x)\left(p - x\frac{dp}{dx}\right) = 0 \quad \ldots \quad \text{(ii)}$$

$\therefore$ The equarion is satisfied when $p^2 + 32x = 0$ . . . (iii)

$$\text{or} \quad p - x\frac{dp}{dx} = 0 \quad \ldots \quad \text{(iv)}$$

From equation (iv), we have $\frac{dp}{p} = \frac{dx}{x}$ or $p = cx$ . . . . (v)

Eliminating p from this equation and the original equation, we have

$$16x^2 + 2c^2x^2y - c^3x^4 = 0$$

or writing $c = 2c'$, we have

$$2 + c'^2y - c'^3x^2 = 0 \quad \ldots \quad \text{(vi)}$$

as the general solution of the equation.

The factor $p^2 + 32x$ of (ii) leading to equation (iii) will not be considered here. It may incidently be noted that elimination of p from (iii) and (i) will lead to the relation between x and y without an arbitrary constant satisfying the equation, and so is a singular solution of the equation.

In what follows we shall only considar the general solutions.

[Hint : Sometimes after coming to the eqt. (v) that is $p = cx$ one is tempted to write this as $\frac{dy}{dx} = cx$ and integrating it, to put the solution as $y = \frac{cx^2}{2} + c$, which is incorrect.]

Example 2 :– Solve $y = (1 + p)x + e^p$ . . . (i)

Differentiatng w. r. t. x, $p = (1+p) + (x + e^p)\frac{dp}{dx}$

$$\therefore \quad -1 = (x + e^p)\frac{dp}{dx} \quad \text{or} \quad \frac{dx}{dp} = -(x + e^p)$$

$$\therefore \quad \frac{dx}{dp} + x = -e^p$$

This is a linear equation in x and p, with the I. F. e^p.

$\therefore$ Using this I. F; we have,

$$xe^p = c - \int e^{2p}\,dp = c - \frac{1}{2}e^{2p}.$$

$$\therefore \quad x = ce^{-p} - \frac{1}{2}e^p \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

Substituting this value of x in equation (i) we have

$$y = (1+p)\left[ce^{-p} - \frac{1}{2}e^p\right] + e^p$$

$$\text{or } y = c\,(1+p)\,e^{-p} + \frac{1}{2}(1-p)\,e^p \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

(i) and (iii) together, with p as a parameter, constitutes the solution of the equation.

[Hint : The above equation is a particular example of the general type $y = f(p)\,x + g(p)$ known as Lagrange's Equation, which is solvable for y, and leads to a differential equation in p and x which is linear in $\frac{dx}{dp}$].

Examples : IV—B

Solve :

Ans.

1. $xp^2 - 2yp + ax = 0.$ $\qquad [\,2cy = c^2x^2 + c\,]$
2. $p^3 + mp^2 = a(y + mx).$

$$\left[\,ax + c = \frac{3}{2}p^2 - mp + m^2\log(p+m)\right.$$
with the given relation]

3. $4y = x^2 + p^2.$ $\qquad \left[\,\log(p-x) = \frac{x}{p-x} + c \text{ with given relation}\,\right]$

4. 4. Solvable for x :— It may sometimes be found convenient to express x in terms of p and y. Let the given equation be arranged as

$$x = F(y, p) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7)$$

Differentiating the equation (7) with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p}\frac{dp}{dy} \qquad \ldots \qquad \ldots \qquad (8)$$

As in equation (8), $\frac{\partial F}{\partial y}, \frac{\partial F}{\partial p}$ are functions of y and p and it is a differential equation of first degree and first order. Integrating, we get

$$f(y, p, c) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Then the elimination of p from (7) and (9) gives the general solution of the given equation. In case p could not be eliminated from (7) and (9), then (7) and (9) together constitute the solution of the equation considering p as a parameter.

Example 1. Solve $x + \frac{p}{\sqrt{1+p^2}} = a.$

We have $\quad x = a - \frac{p}{\sqrt{1+p^2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad$ (i)

Differentiating w. r. t. y,

$$\frac{dx}{dy} = \frac{1}{p} = -\frac{\left[\sqrt{1+p^2} + \frac{p^2}{\sqrt{1+p^2}}\right]\frac{dp}{dy}}{1+p^2}$$

$$\text{or } \frac{1}{p} = -\frac{1}{(1+p^2)^{3/2}} \frac{dp}{dy}$$

$$\text{or } dy + \frac{p}{(1+p^2)^{3/2}}\, dp = 0$$

On integration, $y - \frac{1}{\sqrt{1+p^2}} = -c \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad$ (ii)

To eliminate p from (i) and (ii), we write these equations as

$$(x-a)^2 = \frac{p^2}{1+p^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (a)$$

and

$$(y+c)^2 = \frac{1}{1+p^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (b)$$

Adding (a) and (b) p is eliminated leading to the required general solution

$$(x-a)^2 + (y+c)^2 = 1.$$

Example 2. Solve : $\quad p^2 - 2xyp + 4y^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad$ (i)

Solving for x, $\quad 2x = \frac{p^2}{y} + \frac{4y}{p}.$

Differentiating with respect to y,

$$\frac{2}{p} = \frac{2p}{y}\frac{dp}{dy} - \frac{p^2}{y^2} + 4\left(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}\right)$$

$$\left(p - 2y\frac{dp}{dy}\right)\left(2y^2 - p^3\right)) = 0$$

The factor $(2y^2-p^2)=0$ leads to a singular solution, so omitting it we have

$$p-2y\frac{dp}{dy}=0 \text{ or } \frac{2dp}{p}-\frac{dy}{y}=0$$

Integrating $\quad p=cy^{1/2}$. .. .. (ii)

Eliminating p from (i) and (ii), we have

$$c^2y^{2/2}-2xy\,cy^{1/2}+4y^2=0$$

or $\quad c^2-2cx+4y^{1/2}=0$, or $4y^{1/2}=c(2x-c^2)$

$\therefore \quad 16y=c^2(2x-c^2)^2$

or $\quad 16y=c'(2x-c')^2$ where $c'=c^2$, which is the required solution.

Examples : IV–C

Solve

1. $4x=py(p^2-3)$.

[Ans. $y=c(p^2-4)^{3/20}(p^2+1)^{3/5}$,

$4x=cp(p^2-3)/(p^4-4)^{9/10}(p^2+1)^{3/5}$]

2. $p=\tan\left(x-\dfrac{p}{1+p^2}\right)$.

$$\left[\text{Ans. } \sqrt{\frac{1-c+y}{c-y}}=\tan\left[x-\sqrt{(c-y)-(c-y)^2}\right]\right]$$

3. $ayp^2+(2x-b)p-y=0$. $\qquad$ [Ans. $ac^2+c(2x-b)-y^2=0$.]

4·5. Clairaut's form of the equation :– After considering the three major methods above, we consider here one special form of the equation, which is known as Clairaut's Form. The type of this equation is

$$y=px+f(p) \quad .. \quad .. \quad .. \quad (10)$$

and its solution is obtained easily by the method of "solvable for y".

Differentiating (10) *w. r. t.* x, we get

$$\frac{dp}{dx}=p+[x+f'(p)]\frac{dy}{dx}$$

$$\therefore [x+f'(p)]\frac{dp}{dx}=0 \left[\text{ as } p=\frac{dy}{dx}\right]$$

The factor $[x+f'(p)]=0$ leads to a singular solution and hence omitting it, we have

$$\frac{dp}{dx}=0 \quad \text{i.e. } p=c \quad .. \quad .. \quad .. \quad (11)$$

and eliminating p between (10) and (11), we get

$$y=cx+f(c) \quad .. \quad .. \quad .. \quad (12)$$

Thus the equation has the peculiarity, that the general solution is immediately obtained by writing c for p.

Many problems of differential equations of the first order and higher degree, by substitution of variables, can be reduced to the Clairauts' form, and the solution obtained immediately. The method of substitution is dealt within the next article.

Example 1. $y = px + \sqrt{4 + p^2}$.

The equation being in the Clairaut's form, its primitive is

$$y = cx + \sqrt{4 + c^2}.$$

The example done by method of "solvable for y" will be differentiating w. r. t. x,

$$p = p + x\frac{dp}{dx} + \frac{p}{\sqrt{4 + p^2}}\frac{dp}{dx}$$

$$\text{or} \quad \frac{dp}{dx}\left[x + \frac{p}{\sqrt{4 + p^2}}\right] = 0$$

Taking the factor $\dfrac{dp}{dx} = 0$, we have its solution as $p = c$. So substituting this value of p in original equation, we get for the solution

$y = cx + \sqrt{4 + c^2}$, the same as given above.

Example 2. Solve $p^2(x^2 - 1) - 2pxy + (y^2 - 1) = 0$

The given equation can be written as

$$(p^2x^2 - 2pxy + y^2) = 1 + p^2$$

$$\text{or} \quad (y - px)^2 = 1 + p^2$$

$\therefore y = px \pm \sqrt{1 + p^2}$ which is Clairaut's form and so the required solution is

$$(y - cx + \sqrt{1 + c^2})(y - cx - \sqrt{1 + c^2}) = 0$$

$$\text{or} \quad (y - cx)^2 = 1 + c^2.$$

Examples : IV – D

Solve :– Ans.

1. $y = px + \dfrac{m}{p}$. $\left[y = cx + \dfrac{m}{c}\right]$.
2. $y - x\dfrac{dy}{dx} = e^{dy/dx}$. $\left[y = cx + e^c\right]$.
3. $y = p(x - b) + \dfrac{a}{p}$. $\left[y = c(x - b) + \dfrac{a}{c}\right]$.
4. $p^2x(x-2) + p(2y - 2xy - x + 2) + y^2 + y = 0$. $[(y - cx + 2c)(y - cx + 1) = 0]$
5. $\sin(px - y) = p$ $[y = cx - \sin^{-1}c]$

4·6. Method of Substitution :—In case the given equation cannot be solved for p or y or x, then a substitution of the

variables is necessary. Generally, the substitution will reduce the equation to the Clairaut's Form discussed above, from which the solution can be immediately obtained.

(*a*) Substitution suggested by the nature of the equation.

Example 1. $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0.$

Here the equation cannot be conveniently solved for p, y or x, and so a substitution is necessary.

The derivative of (x^2+y^2) is $2(x+py)$ and that of $x+y$ as $(1+p)$ and so the suggested substitution is

$$x+y=u \text{ and } x^2+y^2=v$$

$$\therefore 1+p=\frac{du}{dx} \text{ and } 2(x+py)=\frac{dv}{dx}.$$

Using these substitutions, the equation becomes

$$v\left(\frac{du}{dx}\right)^2-u\left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right)+\frac{1}{4}\left(\frac{dv}{dx}\right)^2=0$$

$$\text{or } v-u\left(\frac{dv}{du}\right)+\frac{1}{4}\left(\frac{dv}{du}\right)^2=0$$

$$\therefore v=u\left(\frac{dv}{du}\right)-\frac{1}{4}\left(\frac{dv}{du}\right)^2$$ which is Clairaut's form in v and u, and so its solution is

$$v=cu-\frac{1}{4}c^2$$

$$\text{or } (x^2+y^2)=c(x+y)-\frac{1}{4}c^2.$$

Example 1. Solve $p^2\cos^2 y+p\sin x\cos x\cos y-\sin y\cos^2 x=0$

Here it is suggested that we may try the substitution $u=\sin y, v=\sin x$.

$$\therefore p=\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\frac{dv}{dx}=\frac{\cos x}{\cos y}\cdot\frac{du}{dv}.$$

$\therefore$ The equation becomes

$$u=v\frac{du}{dv}+\left(\frac{du}{dv}\right)^2$$. Being Clairaut's form, the solution is

$$u=cv+c^2 \text{ or } \sin y=c\sin x+c^2.$$

(*b*) If the equation be of the type

$$y=nxp+f(x,p)$$

the substitution $x=X^n$ is convenient ; similarly if the equation be of the type

$$y=nxp+f(y,p)$$

then $y^n=Y$ is convenient. No doubt these types of problems can be solved by the method of " solvable for y " but much labour is saved by the substitutions.

Example 3. Solve $y = 3px + 6y^2p^2$.

This is of the type $y = nxp + f\{(y, p)\}$ with $n = 3$; and so we use the substitution $y^3 = Y$. We have then

$$3y^2 \frac{dy}{dx} = \frac{dY}{dx}. \qquad \text{Using } P = \frac{dY}{dx},\ 3y^2p = P$$

$\therefore$ The equation becomes :

$$y = 3x.\frac{P}{3y^2} + 6y^2.\frac{P^2}{9y^4} \text{ or } y^3 = Px + \frac{2}{3}P^2.$$

$\therefore Y = Px + \frac{2}{3}P^2$. Being in Clairaut's form the solution is

$$Y = cx + \frac{2}{3}c^2 \text{ or } y^3 = cx + \frac{2}{3}c^2$$

Example 4. Solve $y = -xp + x^4p^2$

Here the example is of the type $y = nxp + f(x, p)$ with $n = -1$, and the substitution $x = X^{-1}$ is used. We have

$$P = \frac{dy}{dX} = \frac{dy}{dx}\cdot\frac{dx}{dX} = -\frac{p}{X^2} \text{ or } p = -PX^2$$

$\therefore$ The given equation becomes

$$y = -X^{-1}(-PX^2) + X^{-4}.(-PX^2)^2$$

or $\quad y = PX + P^2$, which is the Clairaut's form with the solution

$$y = cX + c^2 \text{ or } y = \frac{c}{x} + c^2$$

(c) *Polar transformation* :–

Polar substitution $x = r\cos\theta, y = r\sin\theta$ is also suggested by the form of the equation. In particular besides the factor $(xdx + ydy)$, $(ydx - xdy)$ etc., if the differential equation involves the factor $\dfrac{y - px}{\sqrt{1+p^2}}$ the polar transformation is convenient. It may by noted here that $\dfrac{y - px}{\sqrt{1+p^2}}$ denotes the length l of the perpendicular from the origin to the tangent at any point (x, y) of the curve, and has its equivalent in polar given by

$$\frac{1}{l^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

Example 5 Solve $y - px = \sqrt{1+p^2}\,(x^2 + y^2)$

The equation can be arranged as

$$\frac{y - px}{\sqrt{1+p^2}} = x^2 + y^2. \text{ or } \left(\frac{y - px}{\sqrt{1+p^2}}\right)^2 = (x^2 + y^2)^2$$

Using the substitution $x = r\cos\theta$, $y = r\sin\theta$, this becomes

$$\frac{r^4}{r^2 + \left(\frac{dr}{d\theta}\right)^2} = r^2$$

$$\text{or } r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1$$

$$\therefore \frac{dr}{\sqrt{1-r^2}} - d\theta = 0.$$

and integating $\quad \sin^{-1} r - \theta = c$

$\therefore$ The required solution is $\sin^{-1}\sqrt{x^2+y^2} - \tan^{-1}\frac{y}{x} = c$.

(d) For problems involving e^{lx} and e^{my}, if k is the H. C. F. of l, m the substitution $X = e^{kx}$ and $Y = e^{ky}$, transforms the equation to the Clairaut's form.

Example 6. Solve $e^{4x}(p-1) + e^{2y}p^2 = 0$.

Let $X = e^{2x}$ and $Y = e^{2y}$

$$\therefore p = \frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{X}{Y}P, \left(P = \frac{dY}{dX}\right)$$

$\therefore$ The equation becomes

$$X^2\left[\frac{X}{Y}P - 1\right] + Y\frac{X^2}{Y^2}P^2 = 0$$

i. e. $\quad Y = XP + P^2$

Hence the solution is

$$Y = cX + c^2 \text{ i. e. } e^{2y} = ce^{2x} + c^2$$

(e) Sometimes a substitution for $\frac{dy}{dx}$ reduces the equation of a second order to a differential equation of the first order. The following example will make this point clear.

Example 7. Solve $\left(\frac{dy}{dx}\right)^2 - y\left(\frac{d^2y}{dx^2}\right) = n\left[\left(\frac{dy}{dx}\right)^2 + a^2\left(\frac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{2}}$

Writing $p = \frac{dy}{dx}$ and noting that we should form a differential equation in p and y, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{dy}{dx} = \frac{dp}{dy}\cdot p.$$

Making use of substitutions in the above equation; we have

$$p^2 - yp\frac{dp}{dy} = n\left[p^2 + a^2p^2\left(\frac{dp}{dy}\right)^2\right]^{1/2}.$$

cancelling p, $\quad p - y\dfrac{dp}{dy} = n\left[1 + a^2\left(\dfrac{dp}{dy}\right)^2\right]^{1/2}$

or $\quad p = y\dfrac{dp}{dy} + n\left[1 + a^2\left(\dfrac{dp}{dy}\right)^2\right]^{1/2}$, which is Clairaut's form in p

and y, with solution

$$p = c_1 y + n\left[1 + a^2 c_1^2\right]^{1/2}$$

We therefore have

$$\frac{dy}{dx} = c_1 y + n\left[1 + a^2 c_1^2\right]^{1/2}$$

$$\text{or} \quad \frac{dy}{c_1 y + n[1 + a^2 c_1^2]^{1/2}} = dx$$

Integrating, $\dfrac{1}{c_1}\log\left\{c_1 y + n[1 + a^2 c_1^2]^{1/2}\right\} = x + \dfrac{1}{c_1}\log c_2$

$$\text{or} \quad c_1 y + n(1 + a^2 c_1^2)^{1/2} = c_2 e^{c_1 x}$$

which is the required solution.

(1) Here we add some more problems involving change of variable.

Example 8. Solve $(px - y)(py + x) = 2p$.

The transformation $y^2 = Y$, $x^2 = X$ gives

$$p = \frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{x}{y}\frac{dY}{dX}$$

$$= \frac{X^{1/2}}{Y^{1/2}}P, \text{ where } P = \frac{dY}{dX}.$$

The equation then becomes

$$\left[\frac{X^{1/2}}{Y^{1/2}}P\,X^{1/2} - Y^{1/2}\right]\left[\frac{X^{1/2}}{Y^{1/2}}P\,Y^{\frac{1}{2}} + X^{\frac{1}{2}}\right] = 2\,\frac{X^{1/2}}{Y^{1/2}}P$$

This simplifies to

$$Y = PX - \frac{2P}{1+P} \quad \text{being Clairaut's form, its solution is}$$

$$Y = cX - \frac{2c}{1+c} \quad \text{or } y^2 = cx^2 - \frac{2c}{1+c}.$$

Example 9. Solve $(xp - y)^2 = p^2 - 2\dfrac{y}{x}p + 1$.

Here we substitute $\dfrac{y}{x} = v$ or $y = vx$

This gives $p = \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} = v + Px$ where $P = \dfrac{dv}{dx}$

$\therefore \quad xp - y = x(v + xP) - vx = x^2P$

and $p^2 - 2\dfrac{y}{x}p + 1 = (v + xP)^2 - 2v(v + xP) + 1 = -v^2 + x^2p^2 + 1$

$\therefore$ The differential equation becomes on subsiitution,

$x^4P^2 = -v^2 + x^2P^2 + 1$ or $P^2x^2(x^2 - 1) = 1 - v^2$

$$\text{or } P = \frac{dv}{dx} = \frac{\sqrt{1-v^2}}{x\sqrt{x^2-1}} \quad \therefore \quad \frac{dv}{\sqrt{1-v^2}} - \frac{dx}{x\sqrt{x^2-1}} = 0$$

Integrating we have for the solution $\sin^{-1} v + \sin^{-1}\dfrac{1}{x} = c$

$$\text{or } \sin^{-1}\frac{y}{x} + \sin^{-1}\frac{1}{x} = c$$

(Note : to evaluate the second integral use $x = \operatorname{cosec} \theta$)

Example 10. Solve $(px + y)^2 = py^2$.

Here we use the transformations $y = Y$, and $xy = X$; the latter amounts to $x = \dfrac{X}{Y}$.

We therefore have, $dy = dY$, and $dx = \dfrac{YdX - XdY}{Y^2}$

and so $p = \dfrac{dy}{dx} = \dfrac{Y^2dY}{YdX - XdY} = \dfrac{Y^2P}{Y - XP}$, where $P = \dfrac{dY}{dX}$

$\therefore$ The given equation becomes

$$\left\{\frac{Y^2P}{Y-XP}\,\frac{X}{Y} + Y\right\}^2 = \frac{Y^2P}{Y-XP}\,Y^2$$

This simplifies to $\qquad Y = PX + \dfrac{1}{P}$

and so has the integral $Y = cX + \dfrac{1}{c}$

$$\text{or } y = cxy + \frac{1}{c}.$$

Example : IV—E

Solve:—	Ans.
1. $e^{3x}(p - 1) + p^3e^{2y} = 0.$	$[e^y = ce^x + c^3]$
2. $y - 2px = \sin(xp^2).$	$[y = 2c\sqrt{x} + \sin c^2]$.
3. $p^3 - 4yxp + 8y^2 = 0.$	$[y = c(x - c)^2]$
4. $x^2p^4 + 2px - y = 0.$	$[(c^2 - y)^2 = 4cx]$

5. $(1+x^2)\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}=0.$ [$y=c_1\tan^{-1}x+c_2$].

6. $2(xp+y)^2=3x^2p.$ [Hint : use $x=X$, $xy=Y$,
Ans. $xy=cx-\frac{3}{8}c^2$]

Examples : IV – F

Solve :—

Ans.

1. $p^2y+2px=y.$ [$y^2=2cx+c^2$].
2. $axyp^2+(x^2-ay^2)p-xy=0.$ [$(x^2+ay^2-c)(y-cx)=0$].
3. $e^{2y}p^3+(e^{2x}+e^{3x})p-e^{3x}=0.$ [$e^y=c(e^x+1)+c^3$].
4. $y=\dfrac{2px}{1+p^2}.$ [$y^2=4c(x-c)$].
5. $p(p+y)=x(x+y).$ [$(2y-x^2-c)(y+x-1-ce^{-x})=0$]
6. $p^2x=py+1$ [$x=cy+c^2$].
7. $x^2p^2-2xyp+(2y^2-x^2)=0.$ $\left[\sin^{-1}\left(\dfrac{y}{x}\right)=\pm\log x+c\right].$
8. $x^2(y-px)=yp^2.$ [Hint: use $x^2=X$, $y^2=Y$, $y^2=cx^2+c^2$]
9. $y-2px=\tan^{-1}(xp^2).$ [$y=2c\sqrt{x}+\tan^{-1}(c^2)$].
10. $4(xp^2+yp)=y^4.$ [$y=4c(cxy-1)$].
11. $y^2\log y=xyp+p^2.$ [$\log y=cx+c^2$].
12. $e^{p-y}=p^2-1.$ [$p(p+1)=c(p-1)e^x$ with given relation].
13. $9(y+xp\log p)=(2+3\log p)p^3.$ [$px=c+\frac{1}{3}p^3$ with given relation].
14. $y=p\tan p+\log\cos p.$ [$x=\tan p+c$ with given relation].
15. $y=(2+p)x+p^2.$ [$x=2(2-p)+ce^{-p}/^2$ $y=8-p^2+(2+p)ce^{-p}/^2$]
16. $p^4-(x+2y+1)p^3+(x+2y+2xy)p^2-2xyp=0.$
[Ans. $(y-c)(y-x-c)(2y-x^2-c)(y-ce^{2x})=0$].
17. $y=2px+y^2p^3.$ [$y^2=2cx+c^3$].
18. $16y^3p^2-4xp+y=0.$ [$y^4=c(x-c)$].
19. $(py+nx)^2=(y^2+nx^2)(1+p^2).$ $\left[y+\sqrt{y^2+nx^2}=cx^{1\pm\sqrt{\frac{n-1}{n}}}\right]$
20. $(y-px)=p(x+py)$ [$y^2=2cx+c^2$].
21. $xy^2p^2-y^3p+a^2x=0.$ [use $x^2=u$, $y^2=v$; $cy^2-c^2x^2=a^2$].
22. $(px-y)(py+x)=h^2p$ $\left[y^2-cx^2+\dfrac{ch^2}{c+1}=0\right].$
23. $16x^2+2p^2y-p^3x=0.$ [$2+c^2y-c^3x^2=0$].
24. $y=2px-p^3y^2+pe^{-2py}$ [$8y^2=8cx+ce^{-c}-c^3$].
25. $y=2px+p^n$ $\left[y=\dfrac{2c}{p}+\dfrac{1-n}{1+n}p^n\right]$

25. Find the curve such that the tangent line at any point P on it bisects the angle between the ordinate at P and the line joining P and the origin.

(Eqt. $y = xp - \frac{x}{p}$. Ans. $c^2x^2 - 2cy - 1 = 0$)

26. The tangent to the curve meets the coordinate axes in A, B and the length AB is constant. Express the statement by means of a differential equation and deduce the form of the curve.

$$\left(\text{Ans. } \frac{(px - y)\sqrt{1 + p^2}}{p} = a \; ; \; y = cx - \frac{ac}{\sqrt{1 + c^2}} \right)$$

27. In a certain curve the tangent at P meets the axes of x in T and PT is always equal to n.OT, n being constant. Express the statement by means of a differential equation. Solve the equation in the case $n = 1$

(Hint : Change to Polar. Ans $r = c \sin \theta$].

28. Find the shape of the reflector such that light coming from a fixed source is reflected in parallel rays.

[The source is taken as origin and x axis parallel to rays then eqt. is $2xp = y - yp^2$].

CHAPER V

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

5·1. Introduction :—A differential equation which contains the differential coefficients and the dependent variable in the first degree, and does not involve the products of the derivatives and the dependent variable is called a linear differential equation. Such an equation of the *n*th order can be represented by

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n y = X \ldots \quad (1)$$

where $P_1, P_2, \ldots P_n$ and X are functions of x only, or constants.

If $P_1, P_2, \ldots P_n$ are independent of x, that is constant, say $a_1, a_2 \ldots a_n$, then the equation is called a linear differential equation with constant coefficients and we represent this equation by

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n y = X \ldots\ldots \quad (2)$$

or using the differential operator D to stand for $\frac{d}{dx}$ (so that $Dy = \frac{dy}{dx}, D^n y = \frac{d^n y}{dx^n}$ etc.) the equation (2) is

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} \ldots + a_n) y = X \ldots \quad (2a)$$

or using f (D) for the polynomial operator $(D^n + a_1 D^{n-1} + \ldots + a_n)$ we write (2) or (2*a*) symbolically as

$$f(D)\, y = X \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2b)$$

We shall confine ourselves in this chapter to the linear differential equation with constant coefficients, the study of which is most important for the solution of Engineering Problems.

5·2. The Differential Operator :— Since we are going to develop the methods of solving the equation (2) by the use

of the differential operator D, it is necessary to be clear about the nature of D.

The symbol D, as referred to above stands for the operation of differentiation, that is $D = \frac{d}{dx}$, so that $Dy = \frac{dy}{dx}$, likewise $D^n y = \frac{d^n y}{dx^n}$ and $(D+a)\,y = \frac{dy}{dx} + ay$.

The differential operator D can be treated much the same as an algebraic quantity, except that it must be remembered that it represents differentiation. Some of the properties of the operator are given below without proof.

$D^n cy = cD^n y$ $\qquad c =$ constant.

$D\,(y+z) = Dy + Dz$ $\qquad$ Distributive law.

$(D-m_1)\,(D-m_2)\,y = (D-m_2)\,(D-m_1)\,y$ $\qquad$ Commutative law.

$= [\,D^2-(m_1+m_2)D+m_1m_2\,]\,y$ $\qquad$ Factor Law

$D^m\,(D^n)\,y = D^{m+n}y$ $\qquad$ Index Law.

We particulaarly note over here the factor law and the commutative law. It means that the polynomial expression in D involving constant coefficients can be factorised by ordinary rules of algebra before differentiation, and the factors may be taken in any order we please. Thus,

$$(D^2+3D-4)y = (D+4)(D-1)\,y = (D-1)(D+4)y = \frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y$$

5·3· Linear differential equation f (D) y = 0 : — Before we consider the method of solving the equation (2) or (2b) that is $f\,(D)\,y = X$, let us first consider the simpler equation, with $X = 0$, or

$$f\,(D)\,y = 0 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3)$$

Since $f\,(D)$ is a polynomial in D of nth degree and D obeys the laws of algebra as stated above, we can in general factorise $f\,(D)$ in n linear factors, and so the equation (3) can be written as

$$(D-m_1)(D-m_2)\;..\;(D-m_r)\;\dots\;(D-m_n)\;y = 0 \;\dots\dots\dots\dots \;(4)$$

where $m_1\; m_2\; ..\; m_r, ..m_n$ are the roots of the algebraic equation

$$f\,(D) = 0\;..\quad ..\quad ..\quad ..\quad \dots\quad (5)$$

The equation (5) is known as an *auxiliary equation*.

Now, the equation (4) will be satisfied by the solution of the equation $(D-m_n)y=0$ that is by $y=c_ne^{m_nx}$.

$\left[(D-m_n)y=0 \text{ is } \frac{dy}{dx}-m_ny=0 \text{ or } \frac{dy}{y}=m_ndx \text{ and so has the solution } y=c_ne^{m_nx}\right]$.

Similarly since the factors in (4) can be taken in any order, the equation will also be satisfied by the solution of each of the equations $(D-m_1)y=0$, $(D-m_2)y=0$ ....etc., that is by $y=c_1e^{m_1x}, y=c_2e^{m_2x}$, ... It can then easily be proved that the sum of these individual solutions, i. e.

$$y=c_1e^{m_1x}+c_2e^{m_2x}+ \dots\dots\dots + c_ne^{m_nx} \dots \quad (6)$$

also satisfies the equation (4), and as it contains n arbitrary constants, and the equation (3) is of the nth order, (6) constitutes the general solution of the equation (3).

Thus the general solution of the equation $f(D)y=0$ is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+\dots\dots\dots+c_ne^{m_nx}$$

where $m_1, m_2, \dots\dots m_n$ are the roots of the auxiliary equation $f(D)=0$.

Example 1. Solve $(2D^2+5D-12)y=0$.

The given equation can be written as

$$(2D-3)(D+4)y=0,$$

and the auxiliary equation is $(2D-3)(D+4)=0$, and has the roots $m_1=\frac{3}{2}$ and $m_2=-4$.

Therefore the solution of the equation is

$$y=c_1e^{3/2x}+c_2e^{-4x}.$$

34. Different cases depending on the nature of the roots of the equation f (D) = 0.

Depending upon the nature of the roots of the auxiliary equation $f(D)=0$, the solution of the differential equation $f(D)y=0$ takes different forms.

(a) *Roots all real and different* :—

Let the n roots $m_1, m_2, \ldots m_n$ of the equation $f(D) = 0$ be all real and distinct. Then the solution of the equation $f(D)\, y = 0$ is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} \quad \ldots \quad \ldots \quad (7)$$

(b) *Case of Multiple roots* :—

Let m_1 be a multiple real root, say repeated twice, of the equation $f(D) = 0$, that is to say $m_1 = m_2$.

Then the corresponding part of the solution (7) degenerates into

$$c_1 e^{m_1 x} + c_2 e^{m_1 x} \text{ or } (c_1 + c_2)\, e^{m_1 x} = c' e^{m_1 x}.$$

Thus the solution (7) in this case will involve only $(n-1)$ arbitrary constants, and will not be the general solution of the equation. We shall therefore investigate this case from the fundamental principle.

When $m_1 = m_2$ then the corresponding part of the equation is

$$(D-m_1)^2 y = 0 \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

or

$$(D-m_1)(D-m_1)\, y = 0 \quad \ldots \quad \ldots \quad (8a)$$

Let

$$(D-m_1)\, y = v \quad \ldots \quad \ldots \quad \ldots \quad (8b)$$

$\therefore$ The equation (8a) becomes

$$(D - m_1)\, v = 0 \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

of which the solution is $v = c_1 e^{m_1 x}$ ... ...(10)

But as $(D-m_1)\, y = v$ $\therefore$ $(D-m_1)\, y = c_1 e^{m_1 x}$ ... (11)

This is a linear equation of first order with I. F. $e^{-m_1 x}$, and so using it, the solution of equation (11) is

$$y e^{-m_1 x} = c_2 + c_1 \int e^{m_1 x} e^{-m_1 x}\, dx$$

that is $y e^{-m_1 x} = c_2 + c_1 x$ or $y = (c_2 + c_1 x)\, e^{m_1 x}$ ... ... (12a)

Thus in this case instead of $c_1 e^{m_1 x} + c_2 e^{m_1 x}$, we get the solution as $(c_2 + c_1 x)\; e^{m_1 x}$ and the general solution will then contain n arbitrary constants.

It can be shown similarly that if m_1 is a three-fold root then the corresponding part in the solution is

$$(c_3 + c_2 x + c_1 x^2)\, e^{m_1 x}$$

$$\boxed{y = (c_r + c_{r-1}x + c_{r-2}x^2 + \ldots\ldots + c_1 x^{r-1})\, e^{m_1 x}} \quad \ldots\ldots(12)$$

Example 1. Solve $(D+1)^3 (D-4)y = 0$.

Here the auxiliary equation is $(D+1)^3 (D-4) = 0$ and the roots are 4 and -1, the latter root being repeated thrice.

Corresponding to the repeated root -1, the part in the solution will be $(c_1 + c_2 x + c_3 x^2)\, e^{-x}$, and the one due to the root 4 will be $c_4 e^{4x}$.

Therefore the general solution of the given equation is

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + c_4 e^{4x}.$$

(c) *Case of Imaginary Roots :*

Since the coefficients of the auxiliary equation $f(D) = 0$ are all real, if at all the roots be imaginary, they will occur in conjugate pairs. Suppose m_1 and m_2 are two such imaginary roots then

$$m_1 = \alpha + i\beta \text{ and } m_2 = \alpha - i\beta$$

The corresponding part of the solution of the equation $f(D)\, y = 0$, then takes the form

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$= e^{\alpha x} [c_1 e^{i\beta x} + c_2 e^{-i\beta x}]$$

$$= e^{\alpha x} [(c_1 + c_2) \cos \beta x + i (c_1 - c_2) \sin \beta x]$$

We can write $c'_1 = c_1 + c_2$ and $c'_2 = i(c_1 - c_2)$ as two new arbitrary constants and so the corresponding part of the solution can be written as

$$\boxed{y = e^{\alpha x} [c'_1 \cos \beta x + c_2' \sin \beta x]} \quad \ldots(13)$$

(d) Case of Repeated Imaginary Roots :—

It may happen that the imaginary roots occurring in conjugate pairs are repeated. Let $m_1 = \alpha + i\beta$ be repeated twice, and $m_2 = \alpha - i\beta$ be also repeated twice.

Then corresponding to m_1 and m_2 we have in the solution

$$y = (c_1 + c_2 x)\, e^{m_1 x} + (c_3 + c_4 x)\, e^{m_2 x}$$

$$= [(c_1 + c_2 x) e^{(\alpha + i\beta x)} + (c_3 + c_4 x)\, e^{(\alpha - i\beta) x}].$$

$$= e^{\alpha x} \left\{ (c_1 + c_2 x) e^{i\beta x} + (c_3 + c_4 x) e^{-i\beta x} \right\}$$

Puting $e^{\pm i\beta x} = (\cos \beta x \pm i \sin \beta x)$, this can be written as

$$\boxed{y = e^{\alpha x} \{(A_1 + A_2 x) \cos \beta x + (A_3 + A_4 x) \sin \beta x\}} \quad (14)$$

where $A_1 = c_1 + c_3, A_2 = c_2 + c_4,\ A_3 = c_1 - ic_3, A_4 = c_2 - ic_4,$

Example 3. Solve $(D^2 - 2D + 2)^2 y = 0$.

Here, referring to the Ex. (2) above it is easy to see that we have the imaginary roots $(1 + i)$ and $(1 - i)$ repeated twice ; and since $\alpha = 1$, and $\beta = 1$ we shall have for the solution of the equation

$$y = e^x \{ (A_1 + A_2 x) \cos x + (A_3 + A_4 x) \sin x \}.$$

We shall solve few more problems here of the type $f(D)y = 0$

Example 4. Obtain the general solution of $(D-3)(D-2)^3 y = 0$.

The auxiliary equation is $(D-3)(D-2)^3 = 0$. The root $D = 3$ appears once, while $D = 2$ appears three times, hence the general solution is

$$y = c_1 e^{3x} + (c_2 + c_3 x + c_4 x^2)\, e^{2x}.$$

Example 5. Solve $(D^3 + 7D^2 + 16D + 10) y = 0$.

Factorization gives $(D + 1)(D^2 + 6D + 10) y = 0$.

With the auxiliary equation $(D + 1)(D^2 + 6D + 10) = 0$, the roots are

$$D = -1,\ -3 + i.\ \text{and}\ -3 - i,$$

and the general solution may be expressed in either of the following forms

$$y = c_1 e^{-x} + c_2 e^{(-3 + i)x} + c_3 e^{(-3 - i)x}$$

$$= c_1 e^{-x} + e^{-3x} (A \cos x + B \sin x).$$

Example 6. Solve $\dfrac{d^4 y}{dx^4} + 4y = 0$.

Here the auxiliary equation is $D^4 + 4 = 0$, which on factorization gives $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ The roots of the D-equation are $-1 + i, -1 - i, 1 + i$ and $1 - i$.

The general solution is therefore

$$y = e^{-x}\,[\,A\cos x + B\sin x\,] + e^{x}\,[C\cos x + D\sin x]$$

Examples : V – A

Solve :–

1. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$ Ans. $[\,y = c_1 + e^{-x}(c_2 + c_3x)]$

2. $(D^4 + 2D^2 + 1)\,y = 0.$ $[\,y = c_1\cos x + c_2\sin x + x\,(c_3\cos x + c_4\sin x)\,]$

3. $(D^3 + D^2 - 2D + 12)\,y = 0.$ $\left[c_1e^{-3x} + e^{x}\left(c_2\cos\sqrt{3}\,x + c_3\sin\sqrt{3}\,x\right)\right]$

4. $(2D^2 - D - 10)\,y = 0.$ $\left[y = c_1e^{\frac{5x}{2}} + c_2e^{-2x}\right]$

5 $(D^6 + 6D^4 + 9D^2)\,y = 0.$

$$\left[y = c_1 + c_2x + (c_3 + c_4x)\sin\sqrt{3}\,x + (c_5 + c_6x)\cos\sqrt{3}\,x\right]$$

6. $(D^2 + 25)\;y = 0.$ $[\,y = c_1\cos 5x + c_2\sin 5x]$

5·5. Linear differential equation f (D) y = X :–

We are now in a position to consider the method of solving the equation $f(D)\,y = X$.

The general solution of the equation $f(D)\,y = X$ can be put down as

$$y = Y_p + Y_c \qquad \ldots \qquad \ldots \qquad \ldots \qquad (15)$$

Y_c is the solution of the given equation with $X = 0$, that is of equation $f(D)\;y = 0$, (which is known as the associated equation) and is called *the complimentary function*. It involves n arbitray constants and is denoted by C. F. We have by definition of Y_c, $f\,(D)\;Y_c = 0$.

Y_p is any function of x, which satisfies the equation $f(D)\,y = X$, so that $f(D)\;Y_p = X$. Y_p is called *the partcular integral* and is denoted by P. I. It does not contain any arbitrary constants. Thus, on substituting $y = Y_p + Y_c$ in $f\;(D)\,y$, we have

$$\begin{aligned} f\,(D)\,[\,Y_p + Y_c\,] &= f(D)\,Y_p + f(D)\,Y_c \\ &= X + 0 \text{ by definitions of } Y_p \text{ and } Y_c \\ &= X \end{aligned}$$

$\therefore\; y = Y_p + Y_c$ satisfies the equation $f\,(\,D\,)\,y = X$ and as it contains n arbitrary constants, is the general (or complete) solution of the equation.

It remains, then to investigate the methods of obtaining the complimentary function, and the particular integral.

5·6. The Complimentary Function :— We have seen in the above article that the C. F. Y_c of the equation $f(D)\, y = X$ is the solution of the equation $f(D)\, y = 0$ and articles (5·3) and (5.4) provide us with the methods of finding the complimentary function.

Example 1. Find the complimentary function of the equation

$$(D^2 + 4D + 3)\, y = e^x. \qquad \dots \quad \dots \quad \dots \quad (16)$$

The complimentary function of the given equation is the solution of the equation

$$(D^2 + 4D + 3)\, y = 0 \qquad \dots \quad \dots \quad \dots \quad \dots \quad (17)$$

The auxiliary equation is $D^2 + 4D + 3 = 0$ or $(D + 3)(D + 1) = 0$ and so the roots are -3 and -1.

Therefore the solution of the equation (17) and the C. F. of the given equation (16) is

$$y_c = c_1 e^{-3x} + c_2 e^{-x}.$$

5·7. The inverse operator $\dfrac{1}{f(D)}$ and the symbolic expression for the particular integral :— Before we take the general methods of finding the Particular Integral, it is necessary to define the inverse operator $\dfrac{1}{f(D)}$, We define $\dfrac{1}{f(D)}\, X$ (where X is a function of x only) as that function of x. which when acted upon by the differential operator $f(D)$ gives X Thus by this definition.

$$f(D) \left\{ \frac{1}{f(D)} \right\} = X \qquad \dots \quad \dots \quad \dots \quad (18)$$

and so $\left\{ \dfrac{1}{f(D)} X \right\}$ satisfies the equation $f(D)\, y = X$ and so is the P. I. of the equation $f(D)\, y = X$.

Thus the P. I. of the equation $f(D)\, y = X$ is symbolically given by

$$Y_p = \frac{1}{f(D)} X \qquad \dots \quad \dots \quad \dots \quad \dots \quad (19)$$

Before going into more detail in the general expression $\frac{1}{f(D)}X$, we shall first take up some simpler expressions involving the inverse operators.

(*a*) Consider first the inverse operator $\frac{1}{D}$. Since D obeys simple laws of algebra(and also from the above definition) we have

$$\left\{ D.\frac{1}{D}\phi(x) \right\} = \phi(x) = D\left\{ \int \phi(x)\,dx \right\}$$

$$\therefore \frac{1}{D}\phi(x) \text{ means } \int \phi(x)\,dx.$$

Similarly as $D^n \left\{ \frac{1}{D^n}\phi(x) \right\} = \phi(x)$, we have

$$\frac{1}{D^n}\phi(x) = \int\int \cdots \int \phi(x)\,(dx)^n.$$

For instance, $\frac{1}{D^3}x^2 = \int\int\int (x^2)\,(dx)^3 = \frac{x^5}{60}$.

(*b*) Next if a be a constant, then

$$\frac{1}{f(D)}[a\,\phi(x)] = a\frac{1}{f(D)}\phi(x)$$

for $f(D)\left\{ a\,\frac{1}{f(D)}\,\phi(x) \right\} = a\,f(D).\ \frac{1}{f(D)}\phi(x) = a\,\phi(x.)$

and operating $\frac{1}{f(D)}$ on the two sides of this, the above result follows.

(*c*) We then consider the inverse operator $\frac{1}{(D-m)}$. By definition of the P. I, $\frac{1}{(D-m)}X$ will be the P.I. of the equation $(D-m)y = X$; i. e. the part in the solution of this equation which does not contain the arbitrary constant. Now this equation

is $\dfrac{dy}{dx} - my = X$, which is a linear equation of the first order with the I. F. e^{-mx} and so has for its solution

$$ye^{-mx} = c + \int Xe^{-mx}\,dx$$

or $\quad y = ce^{mx} + e^{mx}\int X e^{-mx}\,dx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$

In (20) ce^{mx} is the C. F. and $e^{mx}\int Xe^{-mx}\,dx$ is the P. I. Thus the P. I. is given symbolically by

$$\boxed{\frac{1}{D-m}X = e^{mx}\int e^{-mx}X\,dx} \quad \ldots \quad \ldots \quad (21)$$

For instance,

$$\frac{1}{D-2}e^{3x} = e^{2x}\int e^{-2x}e^{3x}\,dx \text{ by (21) above}$$

$$= e^{2x}\int e^{x}\,dx = e^{3x}.$$

5.8. Particular integral $\dfrac{1}{f(D)}X$; the general methods;

We shall now consider how to evaluate the P. I. of the equation

$$f(D)\,y = X \text{ or } \frac{1}{f(D)}X.$$

(*a*) **Method of factors :**

In this method $f(D)$ is factorised into n linear factors so that

By using the formula (21), we can first evaaluate $\frac{1}{D-m_n} X = e^{m_n x} \int e^{-m_n x} X \, dx = X_1$ say, and the result so obtained is then operated upon by $\frac{1}{D-m_{n-1}}$ and so on in succession, until finally the value of $\frac{1}{f(D)} X$ is obtained.

Consider for example $\frac{1}{(D-2)} \cdot \frac{1}{(D-1)} e^{4x}$.

Here we first evaluate $\frac{1}{D-1} e^{4x} = e^{x} \int e^{4x} e^{-x} \, dx = e^{x} \int e^{3x} \, dx = \frac{e^{4x}}{3}$

$$\therefore \frac{1}{D-2}\left\{\frac{1}{D-1} \cdot e^{4x}\right\} = \frac{1}{D-2}\left\{\frac{e^{4x}}{3}\right\} = \frac{1}{3} \cdot \frac{1}{D-2} e^{4x}$$

$$= \frac{1}{3} e^{2x} \int e^{4x} e^{-2x} \, dx = \frac{1}{6} e^{4x}$$

Thus $\frac{1}{(D-2)} \cdot \frac{1}{(D-1)} e^{4x} = \frac{e^{4x}}{6}$.

(*b*) **Method of partial fractions :—**

It is also possible to express $\frac{1}{f(D)}$ into n partial fractions.

Thus

$$\frac{1}{f(D)} = \frac{N_1}{D-m_1} + \frac{N_2}{D-m_2} + \ldots \ldots + \frac{N_n}{D-m_n} \quad \ldots (23)$$

where $N_1, \ldots N_n$ are certain constants.

Hence the P. I. is,

$$\frac{1}{f(D)} X = \left\{ \frac{N_1}{D-m_1} + \frac{N_2}{D-m_2} + \ldots + \frac{N_x}{D-m_n} \right\} X$$

$$= \frac{N_1}{D-m_1} X + \frac{N_2}{D-m_2} X + \ldots + \frac{N_n}{D-m_n} X$$

$$= N_1 e^{m_1 x} \int e^{-m_1 x} X \, dx + N_2 \, e^{m_2 x} \int e^{-m_2 x} X \, dx$$

$$+ \ldots + N_n e^{m_n x} \int e^{-m_n x} X \, dx$$

As an example, consider the previous problem that is

$$\frac{1}{(D-2)(D-1)}e^{4x}=\left\{\frac{1}{D-2}-\frac{1}{D-1}\right\}e^{4x}=\frac{1}{D-2}e^{4x}-\frac{1}{D-1}e^{4x}$$

$$=e^{2x}\int e^{-2x}e^{4x}\,dx-e^{x}\int e^{-x}e^{4x}\,dx$$

$$=e^{2x}\frac{e^{2x}}{2}-e^{x}\frac{e^{3x}}{3}=\frac{e^{4x}}{6}.$$

Example Solve $\dfrac{d^2y}{dx^2}-y=2+5x.$

We shall first find out the C. F. The auxiliary equation is $D^2-1=0$ therefore the roots are 1 and -1, and so the C. F. is given by

$$y_c=c_1e^{x}+c_2e^{-x}.$$

The P. I. of the equation is given by

$$\frac{1}{(D^2-1)}[2+5x]=\frac{1}{2}\left\{\frac{1}{D-1}-\frac{1}{D+1}\right\}[2+5x]$$

$$=\frac{1}{2}\left\{\frac{1}{D-1}[2+5x]-\frac{1}{D+1}[2+5x]\right\}$$

$$=\frac{1}{2}\left\{e^{x}\int e^{-x}(2+5x)\,dx-e^{-x}\int e^{x}(2+5x)\,dx\right\}$$

$$=\frac{1}{2}\Big[-4-10x\Big]=-2-5x$$

Thus the P. I. $y_p=-2-5x$.

And the complete solution of the equation is

$$y=y_c+y_p\quad\text{or}\quad y=c_1e^{x}+c_2e^{-x}-2-5x.$$

5. 9. Particular integral : Short methods : Although the general method given in the above article will always work in the theory, it many a times leads to labourious and difficult integration. Thus in our example in the above article if the right-hand side had been $(2+x^4)$ instead of $(2+5x)$, the integration would have been much tedious. To avoid this, short methods of getting the P. I. without actual integration are developed depending upon the particular form of the function X We shall, in what follows, develop these methods.

(a) $X=e^{ax}$:—

Since $De^{ax}=ae^{ax}$, $D^2e^{ax}=a^2e^{ax}$ and $D^ne^{ax}=a^ne^{ax}$, therefore

$$f(D)\,e^{ax}=f(a)\,e^{ax}$$

Operating by $\frac{1}{f(D)}$ on both sides this equation, we get

$$\frac{1}{f(D)}\left\{f(D)\,e^{ax}\right\} = \frac{1}{f(D)}\left\{f(a)e^{ax}\right\}$$

$$\therefore\ e^{ax} = f(a)\left\{\frac{1}{f(D)}\,e^{ax}\right\}$$

$$\boxed{\therefore\ \frac{1}{f(D)}\,e^{ax} = \frac{1}{f(a)}\,e^{ax},\ \text{provided } f(a) \neq 0} \quad \ldots (24)$$

If $f(a) = 0$, then $(D - a)$ is a factor of $f(D)$, thus this rule fails. Suppose $f(D) = (D-a)\,\phi(D)$ and that $\phi(D)$ does not contain the factor $(D - a)$. Then

$$\frac{1}{f(D)}\,e^{ax} = \frac{1}{(D-a)}\cdot\frac{1}{\phi(D)}\,e^{ax} = \frac{1}{\phi(a)}\cdot\frac{1}{D-a}\,e^{ax}$$

$$= \frac{1}{\phi(a)}\,e^{ax}\int e^{-ax}.\,e^{ax}\,dx,\ \text{[by equation (21)]}$$

$$= \frac{1}{\phi(a)}\,e^{ax}\,x$$

$$\therefore\ \frac{1}{f(D)}\,e^{ax} = \frac{1}{(D-a)}\cdot\frac{1}{\phi(D)}\,e^{ax} = \frac{xe^{ax}}{\phi(a)} \quad \ldots\ldots (25a)$$

Similarly, if $(D - a)^2$ occurs as a factor in $f(D)$, say

$$f(D) = (D - a)^2\,\phi(D),\ \text{then}$$

$$\frac{1}{f(D)}\,e^{ax} = \frac{1}{(D-a)^2}\cdot\frac{1}{\phi(D)}\,e^{ax} = \frac{1}{\phi(a)}\cdot\frac{1}{(D-a)}\cdot\frac{1}{(D-a)}\,e^{ax}$$

$$= \frac{1}{\phi(a)}\cdot\frac{1}{(D-a)}\,xa^{ax}$$

$$= \frac{1}{\phi(a)}\,e^{ax}\int e^{-ax}\ xe^{ax}\,dx = \frac{x^2}{2\,!}\ \frac{e^{ax}}{\phi(a)}$$

and in general it can be proved that if $(D - a)^r$ occurs as a factor in $f(D)$, so that $f(D) = (D-a)^r\,\phi(D)$, then

$$\boxed{\frac{1}{f(D)}\,e^{ax} = \frac{1}{(D-a)^r}\cdot\frac{1}{\phi(D)}\,e^{ax} = \frac{x^r}{r!}\ \frac{e^{ax}}{\phi(a)}} \quad \ldots (25b)$$

Other method

If $f(a) = 0$ then $f(D)$ in general may have a factor $(D-a)^r$ i.e.

$$f(D) = (D-a)^r \phi(D)$$

where $\phi(D)$ does not have a factor $(D-a)$, then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^r} \frac{1}{\phi(D)} e^{ax}$$

$$= \frac{1}{\phi(a)} \frac{1}{(D-a)^r} e^{ax}$$

$$= \frac{1}{\phi(a)} e^{ax} \frac{1}{[(D+a)-a]^r} e^{0x}$$

$$= \frac{1}{\phi(a)} e^{ax} \frac{1}{D^r} 1 \qquad \text{[using result (28)]}$$

$$= \frac{1}{\phi(a)} e^{ax} \frac{x^r}{r!}$$

$$= \frac{x^r}{r!} \cdot \frac{e^{ax}}{\phi(a)}$$

Note :— In operating $\dfrac{1}{(D-a)^r \phi(D)}$ on e^{ax}, it is simpler and advantageous to operate the factor $\phi(D)$ first and then operate on the result thus obtained by $\dfrac{1}{(D-a)^r}$.

Example 1. Find the P. I. of the equation $(D^3 - 2D^2 - 5D + 6)y = e^{4x}$

The P. I. $= \dfrac{1}{D^3 - 2D^2 - 5D + 6} e^{4x}$

$$= \frac{1}{(D-1)(D-3)(D+2)} e^{4x}$$

by equation (24), since here $a = 4$, this is

$$= \frac{1}{(4-1)(4-3)(4+2)} e^{4x}$$

$$= \frac{e^{4x}}{18}.$$

Example 2. Solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x} + 1$.

For the C. F. we have the auxiliary equation $D^3 - 5D^2 + 8D - 4 = 0$ or $(D-1)(D-2)^2 = 0$ with roots 1, 2 and 2.

$\therefore$ The C. F. is $c_1 e^x + (c_2 + c_3 x)e^{2x}$.

The P. I. is $\frac{1}{(D-1)(D-2)^2}\left[e^{2x}+2e^{x}+3e^{-x}+2\right]$

$$=\frac{1}{(D-1)(D-2)^2}e^{2x}+2.\frac{1}{(D-1)(D-2)^2}e^{x}+3.\frac{1}{(D-1)(D-2)^2}e^{-x}$$

$$+\frac{1}{(D-1)(D-2)^2}.2$$

Let us evaluate each of these separately.

$$\frac{1}{(D-1)(D-2)^2}e^{2x}=\frac{1}{(2-1)}.\frac{1}{(D-2)^2}e^{2x}$$

$$=\frac{x^2}{2!}e^{2x} \qquad \text{by equation (25b).}$$

2. $\frac{1}{(D-1)(D-2)^2}e^{x}=2.\frac{1}{(D-1)}.\frac{1}{(1-2)^2}e^{x}=2.xe^{x}$

3. $\frac{1}{(D-1)(D-2)^2}e^{-x}=3.\frac{1}{(-1-1)(-1-2)^2}e^{-x}=-\frac{e^{-x}}{6}$

and $\frac{1}{(D-1)(D-2)^2}.2=\frac{1}{(D-1)(D-2)^2}2e^{0x}$

$$=2.\frac{1}{(0-1)\,0-2)^2}e^{0x}=\frac{2}{-4}=-\frac{1}{2}$$

$$\therefore \text{ The P. I.}=\frac{x^2e^{2x}}{2}+2xe^{x}-\frac{e^{-x}}{6}-\frac{1}{2}$$

The complete solution = C. F. + P. I. therefore

$$y=c_1e^{x}+(c_2+c_3x)e^{2x}+\frac{x^2e^{2x}}{2}+2xe^{x}-\frac{e^{-x}}{6}-\frac{1}{2}.$$

(b) X = sin (ax + b) or cos (ax+b) :—

Now $D\sin(ax+b)=a\cos(ax+b)$

$$D^2\sin(ax+b)=(-a^2)\sin(ax+b)$$

and in general $(D^2)^r\sin(ax+b)=(-a^2)^r\sin(ax+b)$.

Since $f(D^2)$ is a polynomial expression in D^2 so we have

$$f(D^2)\sin(ax+b)=f(-a^2)\sin(ax+b) \qquad \text{.. (26)}$$

Operating by $\frac{1}{f(D^2)}$ on both sides of equation (26) and adjusting the terms, we have

$$\boxed{\frac{1}{f(D^2)}\sin(ax+b)=\frac{1}{f(-a^2)}\sin(ax+b)} \qquad \text{.. (27a)}$$

and it can similarly be proved that

$$\boxed{\frac{1}{f(D^2)}\cos(ax+b)=\frac{1}{f(-a^2)}\cos(ax+b)} \quad \ldots (27b)$$

It may be noted here that we can replace D^2 by $(-a^2)$ at any stage of the evaluation of the P. I.

If $(D^2+a^2)^r$ occurs as a factor in $f(D^2)$, then the above method fails as $f(-a^2)=0$, and we investigate this case as follows :—

Since $e^{i(ax+b)}=\cos(ax+b)+i\sin(ax+b)$.

$$\therefore \frac{1}{(D^2+a^2)^r}\sin(ax+b)=\text{Imaginary part of}\frac{1}{(D^2+a^2)^r}e^{i(ax+b)}$$

$$=\text{Imaginary part of } e^{ib}\frac{1}{(D-ia)^r(D+ia)^r}e^{iax}$$

$$= \quad " \quad " \quad e^{ib}\frac{1}{(D-ia)^r}\cdot\frac{1}{(2ia)^r}e^{iax}$$

[By Eqt. (24)]

$$= \quad " \quad " \text{ of } \frac{e^{ib}}{(2ia)^r}\cdot\frac{1}{(D-ia)^r}e^{iax}$$

$$= \quad " \quad " \text{ of } \frac{e^{ib}}{(2ia)^r}\cdot\frac{x^r}{r!}e^{iax}$$

$$= \quad " \quad " \text{ of } \frac{(-1)^r x^r}{(2a)^r\, r!}e^{i(ax+b)}\, i^r$$

[By Eqt. (25b)]

$$= \quad " \quad " \text{ of } \frac{(-1)^r x^r}{(2a)^r\, r!}e^{i\left(ax+b+\frac{\pi r}{2}\right)}$$

$$\left[\text{as } i^r=e^{i\frac{\pi r}{2}}\right]$$

Hence

$$\frac{1}{(D^2+a^2)^r}\sin(ax+b)=\frac{(-1)^r x^r}{(2a)^r\, r!}\sin\left(ax+b+\frac{\pi r}{2}\right)$$

Similarly

$$\frac{1}{(D^2+a^2)^r}\cos(ax+b)=\frac{(-1)^r x^r}{(2a)^r\, r!}\cos\left(ax+b+\frac{\pi r}{2}\right) \quad (27c)$$

Example 3. Find the P. I. of the equation $(D^3+D^2+D+1)y=\sin 2x$

$$\text{P. I} = \frac{1}{D^3+D^2+D+1}\sin 2x$$

$$= \frac{1}{D^2\cdot D+D^2+D+1}\sin 2x \text{ (Note this step)}$$

$$= \frac{1}{(-4)\,D-4+D+1}\sin 2x, \text{ putting } D^2 = -(2^2) = -4$$

$$= \frac{1}{-3D-3}\sin 2x$$

$$= -\frac{1}{3}\,\frac{D-1}{(D-1)(D+1)}\sin 2x \qquad \text{(Note this step)}$$

$$= -\frac{1}{3}\,(D-1).\ \frac{1}{D^2-1}\sin 2x$$

$$= -\frac{1}{3}(D-1).\ \frac{1}{(-4)-1}\sin 2x, \text{ putting } D^2 = -4$$

$$= \frac{1}{15}\,(D-1)\ \sin 2x = \frac{1}{15}\left[\,2\cos 2x-\sin 2x\,\right]$$

Thus the P. I. $= \dfrac{1}{15}\left[\,2\cos 2x-\sin 2x\,\right]$

9 **Example 4.** Solve $\dfrac{d^2y}{dx^2}+a^2y=\cos ax.$

The C. F. is given by $D^2+a^2=0$ or $D=+ia$ and $-ia$

$\therefore$ C. F. is $c_1\cos ax+c_2\sin ax.$

$$\text{The P. I.} = \frac{1}{D^2+a^2}\cos ax.$$

Here the rule $\dfrac{1}{f(D^2)}\cos ax=\dfrac{1}{f(-a^2)}\cos ax$ fails and so treating $\cos a$. as the real part (R.P.) of e^{iax}, we get

$$\text{P. I.} = \frac{1}{D^2+a^2}\cos ax = \text{R. P. of}\ \frac{1}{D+ia}.\ \frac{1}{D-ia}e^{iax}$$

$$= \text{R. P. of}\ \frac{1}{2ia}\cdot\frac{1}{(D-ia)}e^{iax}$$

$$= \text{R. P. of}\ \frac{1}{2ia}.\ xe^{iax}, \text{ by equation (25b)}$$

$$= \text{R. P. of}\ \frac{-i}{2a}\,xe^{iax}$$

$$= \text{R. P. of}\ \frac{-i}{2a}\,x\left[\cos ax+i\sin ax\right]$$

$$= \frac{x\sin ax}{2a}.$$

By using equation (27c), we get

$$\frac{1}{D^2+a^2}\cos ax = \frac{(-1)x}{(2a)1\,!}\cos\left(ax+\frac{\pi}{2}\right) = \frac{x}{2a}\sin ax$$

Alter :— P. I. $= \dfrac{1}{D^2+a^2}\cos ax$

$$= \lim_{m\to a}\frac{1}{D^2+a^2}\cos mx$$

$$= \lim_{m\to a}\frac{\cos mx}{-m^2+a^2}$$

$$= \frac{x}{2a}\sin ax$$

$\therefore$ The complete solution of the equation is

$$y = c_1\cos ax + c_2\sin ax + \frac{x\sin ax}{2a}.$$

Example 5. Solve $\dfrac{d^2y}{dx^2} + y = \sin x.\sin 2x$

The C. F. obtained from $D^2+1=0$ is
$$c_1\cos x + c_2\sin x$$

For P. I. we have

$$y = \frac{1}{D^2+1}\sin x.\sin 2x$$

$$= \frac{1}{2}\frac{1}{D^2+1}\left[\cos x - \cos 3x\right]$$

$$= \frac{1}{2}\frac{1}{D^2+1}\cos x - \frac{1}{2}\frac{1}{D^2+1}\cos 3x$$

$$= \frac{1}{2}\cdot\frac{x}{2}\sin x - \frac{1}{2}\left(-\frac{1}{8}\right)\cos 3x$$

Hence the solution is

$$y = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$$

(c) X = Sinh (ax + b) or Cosh (ax + b) :—

It can be proved very easily that

$$\boxed{\begin{aligned}\frac{1}{f(D^2)}\sinh(ax+b) &= \frac{1}{f(a^2)}\sinh(ax+b)\\ \text{and}\ \frac{1}{f(D^2)}\cosh(ax+b) &= \frac{1}{f(a^2)}\cosh(ax+b)\end{aligned}} \quad ..(27d)$$

Example 6. Solve $(D^3 + 3D)\,y = \cosh 2x$.

For the C. F. $\quad D^3 + 3D = 0 \;\therefore\; D = 0 \text{ or } \pm i\sqrt{3}$

Therefore C. F. is $c_1 + c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x$

$$\text{The P. I.} = \frac{1}{D^3 + 3D}\cosh 2x = \frac{1}{D^2 D + 3D}\cosh 2x$$

$$\frac{1}{4D + 3D}\cosh 2x = \frac{1}{7}\cdot\frac{1}{D}\cosh 2x = \frac{1}{7}\int \cosh 2x\,dx = \frac{1}{14}\sinh 2x$$

$\therefore$ The complete solution is

$$y = c_1 + c_1 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x + \frac{1}{14}\sinh 2x.$$

(*Alternatively in evaluating the P. I. of terms involving hyperbolic functions their exponential equivalents may be used*).

(d) $X = x^m$:—

If X contains powers of x, such as x^m etc. then the method of procedure to evaluate $\frac{1}{f(D)}\,x^m$ is to write it as $[f(D)]^{-1}\,x^m$, expand $[f(D)]^{-1}$ in ascending powers of D upto the mth power, and evaluate the expression through differentiation of x^m. The derivatives of order higher than m need not be considered. This is illustrated in the following problem :—

Example 7. *Find the P. I. of the equation* $(D^2-2D+4)\,y = 3x^2-5x+2$

$$\text{The P. I.} = \frac{1}{D^2-2D+4}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1 - \frac{D}{2} + \frac{D^2}{4}\right\}^{-1}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1 - \left(\frac{D}{2} - \frac{D^2}{4}\right)\right\}^{-1}(3x^2-5x+2)$$

Expanding by the Binomial theorem, upto the term in D^2 as the highest degree term is x^2

$$= \frac{1}{4}\left\{1 + \frac{D}{2} + \frac{D^2}{4}\ldots\right\}(3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[3x^2 - 5x + 2 + 3x - \frac{5}{2} + \frac{3}{2}\right]$$

$$= \frac{1}{4}\left[3x^2 - 2x + 1\right]$$

Thus the P. I. is $\frac{1}{4}\left(3x^2 - 2x + 1\right)$.

(e) $X = e^{ax}V$, where V is a function of x :—

$$D\, e^{ax}V = e^{ax}DV + a e^{ax}V = e^{ax}(D+a)V$$

$$D^2 e^{ax}V = e^{ax}D^2V + 2a e^{ax}DV + a^2 e^{ax}V = e^{ax}(D+a)^2V$$

and in general,

$$D^n e^{ax}V = e^{ax}(D+a)^nV \quad \ldots \quad \ldots \quad \ldots (28a)$$

Hence,

$$f(D)\, e^{ax}V = e^{ax} f(D+a)V \quad \ldots \quad \ldots \quad \ldots (28b)$$

Now let $f(D+a)V = V_1$, i. e. $V = \dfrac{1}{f(D+a)} V_1$

Substituting for V in (28b), we get

$$f(D)\, e^{ax}. \frac{1}{f(D+a)} V_1 = e^{ax}V_1$$

Operating both sides by $\dfrac{1}{f(D)}$, we obtain

$$e^{ax} \frac{1}{f(D+a)} V_1 = \frac{1}{f(D)} e^{ax} V_1$$

Now since V_1 in any function of x, we have the rule

$$\frac{1}{f(D)} e^{ax}\, V = e^{ax} \frac{1}{f(D+a)} V \quad . \quad \ldots \quad \ldots (28)$$

Example 8. Solve $(D^2-4)\,y = x^2 e^{3x}$.

The C. F. is given by $c_1 e^{2x} + c_2 e^{-2x}$

$$\text{The P. I.} = \frac{1}{(D+2)(D-2)} x^2 e^{3x}$$

$$= e^{3x} \frac{1}{[(D+3)+2][(D+3)-2]} x^2 \text{ by } (28).$$

$$= e^{3x} \frac{1}{D^2+6D+5} x^2$$

$$= \frac{e^{3x}}{5}\left[1 + \frac{6}{5}D + \frac{D^2}{5}\right]^{-1} x^2$$

$$= \frac{e^{3x}}{5}\left[1 - \frac{6}{5}D + \frac{31}{25}D^2 \ldots.\right] x^2$$

$$= \frac{e^{3x}}{5}\left[x^2 - \frac{12}{5}x + \frac{62}{25}\right].$$

$$= \frac{e^{3x}}{125}\left[25x^2 - 60x + 62\right]$$

The general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + \;+\; \frac{e^{2x}}{125}\,[\,25x^2 - 60x + 62\,]$$

(f) $X = xV$, where V is a function of x :—

We have through successive differentiation,

$$\begin{aligned}
D\ (xV) &= xDV + V \\
D^2\ (xV) &= xD^2V + 2DV \\
D^3\ (xV) &= xD^3V + 3D^2V \\
\text{and so, } D^n\ (xV) &= xD^nV + nD^{n-1}V \\
\text{or } D^n\ [xV] &= xD^nV + \frac{d}{dD}[D^n]\ V \quad \ldots [29a]
\end{aligned}$$

From this last equation we have in general, since $f\,[D]$ is a polynomical expression in D,

$$f\ (D)\ [\,xV\,] = xf\ (D)\ V + f'\ (D)\ V \qquad \ldots \quad (29b)$$

$$\text{where } f'\ (D) = \frac{d}{dD}\ f\,(\,D\,).$$

Putting $f\,(D)\ V = V_1$, so that $V = \frac{1}{f\,(D)} V_1$ in this last equation, we get

$$f\,(D)\left[x\ \frac{1}{f\,(D)} V_1\right] = x\ V_1 + f'\ (D)\ \frac{1}{f\,(D)} V_1 \quad \ldots \quad (29c)$$

Operating on both sides of this equation by the operator $\frac{1}{f\,(D)}$, we get

$$x.\ \frac{1}{f\,(D)}\ V_1 = \frac{1}{f\,(D)}\ [xV_1] + \frac{1}{f\,(D)} f'(D)\ \frac{1}{f\,(D)} V_1$$

and adjusting the terms, we have

$$\frac{1}{f\,(D)}\left\{\,xV_1\,\right\} = \left\{\,x - \frac{1}{f\,(D)}.\ f'(D)\,\right\}\frac{1}{f\,(D)}\ V_1$$

and since V_1 is any function of x. we have

$$\boxed{\frac{1}{f\,(D)}\ [xV] = \left\{x - \frac{1}{f\,(D)}.f'\ (D)\right\}\ \frac{1}{f\,(D)} V} \quad \ldots (29)$$

If $X = x^2 V$ or in general $x^n V$, we can use the formula (29) successively to evaluate the P. I. $\dfrac{1}{f(D)}[x^n V]$. *It may be noted over here, however, that in all cases like $X = x^n V$ it is more convenient to use the formula (28), that is one for $\dfrac{1}{f(D)} e^{ax} V$ as is illustrated below, and avoid the necessity of remembering the rather cumbersome formula* (29).

Example 9. Solve $(D^2 + 3D + 2)y = x \sin 2x$.

For the C. F. we have $(D + 2)(D + 1) = 0$ or $D = -2$ or -1

$\therefore$ The C. F. is $c_1 e^{-2x} + c_2 e^{-x}$

$$\text{The P. I.} = \frac{1}{D^2 + 3D + 2}(x \sin 2x)$$

Here $f(D) = D^2 + 3D + 2$ and $f'(D) = 2D + 3$ and $V = \sin 2x$ Therefore using the formula (29), we get

$$\frac{1}{D^2+3D+2}(x \sin 2x) = \left\{x - \frac{2D+3}{D^2 + 3D + 2}\right\} \cdot \frac{1}{D^2+3D+2} \sin 2x$$

$$= x . \frac{1}{D^2 + 3D + 2} \sin 2x - \frac{2D+3}{(D^2+3D+2)^2} \sin 2x$$

$$= x . \frac{1}{3D-2} \sin 2x - \frac{2D+3}{(3D-2)^2} \sin 2x \quad (\text{putting } D^2 = -4)$$

$$= x . \frac{3D+2}{9D^2-4} \sin 2x - \frac{2D+3}{9D^2 - 12D+4} \sin 2x$$

$$= x . \frac{3D + 2}{-40} \sin 2x - \frac{2D + 3}{-12D - 32} \sin 2x$$

$$= \frac{-x}{40}[6 \cos 2x + 2 \sin 2x] + \frac{1}{4} \frac{(2D + 3)(3D - 8)}{(3D + 8)(3D - 8)} \sin 2x$$

$$= \frac{x}{-20}[3 \cos 2x + \sin 2x] + \frac{1}{4} . \frac{(6D^2 - 7D - 24)}{9D^2 - 64} \sin 2x$$

$$= -\frac{x}{20}[3 \cos 2x + \sin 2x] + \frac{(7D + 48)}{400} \sin 2x$$

$$= -\frac{x}{20}[3 \cos 2x + \sin 2x] + \frac{14 \cos 2x + 48 \sin 2x}{400}$$

$$= -\frac{x}{20}[3 \cos 2x + \sin 2x] + \frac{7 \cos 2x + 24 \sin 2x}{200}$$

$\therefore$ The general solution of the equation is

$$y = c_1 e^{-2x} + c_2 e^{-x} - \left(\frac{30x-7}{200}\right) \cos 2x - \left(\frac{5x - 12}{100}\right) \sin 2x.$$

Aliter : To find the P. I. in the above example we use the fact that $e^{i2x} = \cos 2x + i \sin 2x$, so that $\sin 2x$ is the imaginary part (denoted here after by I. P.) of e^{i2x}. Thus

$$\text{The P. I.} = \frac{1}{D^2+3D+2}\, x \sin 2x$$

$$= \text{I. P. of } \frac{1}{D^2+3D+2}\, x e^{i2x}$$

$$= \text{I. P. of } e^{i2x} \cdot \frac{1}{(D+2i)^2+3(D+2i)+2}\, x \quad \text{by formula (28)}$$

$$= \text{I. P. of } e^{i2x}\, \frac{1}{D^2+D(3+4i)+(6i-2)}\, x$$

$$= \text{I. P. of } \frac{e^{i2x}}{6i-2}\left\{1+\frac{3+4i}{6i-2}D+\frac{D^2}{6i-2}\right\}^{-1} x$$

$$= \text{I. P. of } \frac{e^{i2x}}{6i-2}\left\{1-\frac{3+4i}{6i-2}D \ldots\right\} x$$

$$= \text{I. P. of } \frac{e^{i2x}(6i+2)}{-40}\left\{x-\frac{3+4i}{6i-2}\right\}$$

$$= \text{I. P. of } \frac{\cos 2x + i\sin 2x}{-20}\left[x(3i+1)-\frac{(3+4i)(3i+1)(6i+2)}{-40}\right]$$

$$= \text{I. P. of } \frac{\cos 2x + i\sin 2x}{-20}\left[x(3i+1)-\frac{12}{5}--\right]$$

$$= \text{I. P. of } \frac{\cos 2x + i\sin 2x}{-20}\left[i\left(3x-\frac{7}{10}\right)+\left(x-\frac{12}{5}\right)\right]$$

$$= \frac{3x-\frac{7}{10}}{-20}\cos 2x + \frac{x-\frac{12}{5}}{-20}\sin 2x$$

$$= -\frac{30x-7}{200}\cos 2x - \frac{5x-12}{100}\sin 2x$$

which is the same as above.

This method is specially convenient when X is of the form $x^n \begin{matrix}\sin\\ \cos\end{matrix}(ax+b)$.

We give below the summary of the important formulae short methods of finding the P. I. proved above for easy reference.

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, \text{ if } f(a) \neq 0$$

$$\frac{1}{f(D^2)}\begin{matrix}\sin\\ \cos\end{matrix}(ax+b) = \frac{1}{f(-a^2)}\begin{matrix}\sin\\ \cos\end{matrix}(ax+b), \text{ if } f(-a^2) \neq 0$$

$$\frac{1}{f(D^2)}\begin{matrix}\sinh\\ \cosh\end{matrix}(ax+b) = \frac{1}{f(a^2)}\begin{matrix}\sinh\\ \cosh\end{matrix}(ax+b), \text{ if } f(a^2) \neq 0$$

$$\frac{1}{f(D)}x^m = \left\{f(D)\right\}^{-1}x^m, \left\{f(D)\right\}^{-1} \text{ expanded in ascending powers of D.}$$

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$$

$$\frac{1}{f(D)}[xV] = \left\{x - \frac{f'(D)}{f(D)}\right\}\frac{1}{f(D)}V$$

Examples : V—B

Solve — Ans.

1. $\dfrac{d^2y}{dx^2} + 4y = \sin 3x + e^x + x^2$

$$\left[y = c_1\cos(2x+c_2) + \frac{e^x - \sin 3x}{5} + \frac{x^2}{4} - \frac{1}{8}\right]$$

2. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2\cos x.$

$$\left[y = (a+bx)\sin x + (c+dx)\cos x + \frac{x^3\sin x}{12} + \frac{9x^2 - x^4}{48}\cos x\right]$$

3. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}(1+x).$

$$\left[y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} - \frac{e^{2x}}{12}\left(x + \frac{17}{12}\right)\right]$$

4. $\dfrac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}.$

$$\left[y = c_1e^{-x} + c_2e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x + c_3\right) + 3 + \frac{5}{9}e^{2x} + \frac{xe^{-x}}{3}\right]$$

5. $\dfrac{d^2y}{dx^2} + 4y = x\sin x.$

$$\left[y = c_1\cos(2x+c_2) + \frac{x\sin x}{3} - \frac{2}{9}\cos x\right]$$

6. $\dfrac{d^2y}{dx^2} - y = x^2\sin 3x.$ (Hint : use $\sin 3x$ as I. P. of e^{3ix})

$$\left[y = c_1 e^{2x} + c_2 e^{-x} - \frac{25x^2 - 13}{250} \sin 3x - \frac{3x}{25} \cos 3x \right]$$

7. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$.

$$\left[\textbf{Ans. } y = c_1 e^{-x} + c_2 e^{-2x} + e^{-3x} + e^{-2x} e^{e^x} \right]$$

5. 10. Particular integral : Other methods :— In general when X is of the form given in article (5·9), short methods can be used, but if X be of any other form, then one has to make the use of the general method, i. e. the method of factors or partial fractions. But this method involves laborious integration and in such cases the two additional methods given below can be used conveniently. These methods are explained through the problems solved below.

[*a*] *Method of Variation of Parameter* :

We obtain the C. F. of the given equation f [D] y=X which involves arbitrary constants. The P. I. is then formed out of this C. F. by considering the arbitrary constants as some unknown functions of x, which are determined so that the P. I. satisfies the differential equation f (D) y = X. Additional conditional equations are formed in the process of differentiation, for the convenience of applying the method. This method is called the method of variation of parameters and will be better understood with the problem solved below.

Example 1. Solve $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$. .. .. (30)

Here the C. F. is $c_1 \cos ax + c_2 \sin ax$ and the P. I. is $\dfrac{1}{D^2 + a^2} \sec ax$. We have no short-method to evaluate it, and so the recourse is to be made to the general method. But as can be seen, this will lead to cumbersome integration and so we shall use the method of variation of parameters.

We consider the arbitray constants c_1 and c_2 in the C. F. as unknown functions of x, say $u(x)$ and $v(x)$ and put down the P. I. as

$$y = u \cos ax + v \sin ax \quad .. \; .. \; .. \; .. \quad (31)$$

To determine two unknown functions u and v we need two equations of which should be that (31) must satisfy the given equation (30). The other conditional equation is at our disposal, and we shall choose it as will be convenient to solve the problem,

Differentiating (31), we have

Thus the P. I. is $= e^x \cdot \dfrac{\sin 2x + \cos 2x}{8}$ and the complete solution is

$$y = c_1 e^x + c_2 e^{-x} - e^x \cdot \frac{\sin 2x + \cos 2x}{8}$$

Examples : V—C

Solve by the method of variation of parameters :

1. $(D^2+4)y = 4\sec^2 2x$.
 [ans. $y = c_1\cos 2x + c_2\sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$]
2. $(D^2-1)y = (1+e^{-x})^{-2}$.
 [ans. $y = c_1 e^x + c_2 e^{-x} - 1 + e^{-x}\log(1+e^x)$]
3. $(D^3+D)\ y = \operatorname{cosec} x$.
 [ans. $y = c_1 + c_2 \cos x + c_3 \sin x - \log(\operatorname{cosec} x + \cot x) - \cos x \log \sin x - x \sin x$]
4. $(D^2-1)\ y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$.
 [ans. $y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x})$]

Solve by the method of undetermined multipliers :

5. $(D^2-2D)\,y = e^x \sin x$. $\left[\text{ans. } y = c_1 + c_2 e^{2x} - \dfrac{1}{2} e^x \sin x\right]$
6. $(D^2-2D+3)\,y = x^3 + \sin x$.
 [Ans. $y = e^x(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x) + \dfrac{1}{27}(9x^3 + 18x^2 + 6x - 8) + \dfrac{1}{4}(\sin x + \cos x)$]

5·11. Differential Equations reducible to the Linear Differential Equations with constant coefficients :— There are two types of differential equations which can be reduced to the linear differential equations with constant coefficients by convenient substitutions, which we shall study in this article.

(*a*) *The Cauchy Linear Equation* (*Homogeneous equation*) :— An equation of this type is given by

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n y = X. \quad \ldots(4)$$

where $P_1, P_2, \ldots \ldots \ldots P_n$ are constants.

This can be reduced to the linear differential equation with constant coefficients by the substitution

$$z = \log x \text{ or } x = e^z.$$

For then $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \dfrac{1}{x} \cdot \dfrac{dy}{dz}$ or $x\dfrac{dy}{dx} = \dfrac{dy}{dz} = Dy$

using D for $\frac{d}{dz}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}.\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\therefore\ x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} = D(D-1)\,y.$$

Similarly it can be proved that

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y \text{ etc. so that}$$

$$x^r\frac{d^r}{dx^r} = D(D-1)(D-2)\ldots(D-r+1) \quad \ldots (42)$$

where $D = \frac{d}{dz}$

Thus making these substitutions the equation (41) is reduced to the linear form. An example will make this clear.

Example 12. Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$.

Substituting $z = \log x$, that is $x = e^z$, and using D for $\frac{d}{dz}$, the equation becomes

$$[D(D-1) - 2D - 4]\,y = e^{2z} + 2z$$

or $\quad [D^2 - 3D - 4]\,y = e^{2z} + 2z$

which is a linear equation with constant coefficients in y and z. For the C. F. we have $(D-4)(D+1) = 0$ i. e. $D = 4$ or -1

$\therefore$ The C. F. is $c_1e^{-z} + c_2e^{4z}$.

$$\text{The P. I.} = \frac{1}{D^2 - 3D - 4}[e^{2z} + 2z]$$

$$= \frac{1}{D^2 - 3D - 4}e^{2z} + 2\frac{1}{D^2 - 3D - 4}z$$

$$= \frac{1}{4-6-4}e^{2z} - \frac{1}{2}\left(1 + \frac{3D}{4} - \frac{D^2}{4}\right)^{-1}z$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2}\left(1 - \frac{3D}{4}\ldots\ldots\right)z$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2}\left(z - \frac{3}{4}\right)$$

$\therefore$ The solution of the y, z equation is

$$y = c_1 e^{-z} + c_2 e^{4z} - \frac{e^{2z}}{6} - \frac{z}{2} + \frac{3}{8}$$

or in terms of y, x the required solution is

$$y = c_1 x^{-1} + c_2 x^4 - \frac{x^2}{6} - \frac{1}{2}\log x + \frac{3}{8}.$$

(b) *The Legendre Linear Equation* :—

This equation is of the type

$$(a + bx)^n \frac{d^n y}{dx^n} + P_1 (a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = X \quad (4)$$

where $P_1, P_2, \ldots\ldots P_n$ are constants.

Here the substitution $t = a + bx$ will reduce this equation to the Cauchy's form developed above and a further substitution $t = e^z$ reduces it to the linear differential equation with constant coefficients. The substitution totally amounts to putting $a + bx = e^z$.

The example given below will make this clear.

Example 13. Solve $(x + a)^2 \dfrac{d^2y}{dx^2} - 4(x + a)\dfrac{dy}{dx} + 6y = x$

We first set $x + a = t$, so that $\dfrac{dx}{dt} = 1$ and the equation in y and t becomes,

$$t^2 \frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = t - a, \quad \ldots \quad (6)$$

which is of the Cauchy's form.

Further substitution $t = e^z$ that is $z = \log t$ reduces the equation (6) to

$$[D(D - 1) - 4D + 6]\, y = e^z - a, \quad \text{where } D = \frac{d}{dz}$$

or

$$[D^2 - 5D + 6]\, y = e^z - a \quad \ldots \quad (7)$$

which is a linear equation with constant coefficients in y and z, and we have for its solution from previous methods,

$$y = c_1 e^{2z} + c_2 e^{3z} + \frac{e^z}{2} - \frac{a}{6}$$

or since $e^z = t = x + a$, the required solution is

$$y = c_1 (x + a)^2 + c_2 (x + a)^3 + \frac{3x + 2a}{6}$$

Examples : V—D

Solve

1. $(x^3D^3 + x^2D^2 - 2)\, y = x + \dfrac{1}{x^3}$.

$$\left[\text{Ans. } y = c_1 x^2 + c_2 \cos(\log x) + c_3 \sin(\log x) - \frac{x}{2} - \frac{1}{50x^2}\right].$$

2. $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = 3x - 7.$

$$\text{Ans. } y = c_1 x + x^{-1/2}\left[c_2 \sin\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \cos\left(\frac{\sqrt{3}}{2}\log x\right)\right] + x \log x + 7$$

3. $(x^2D^2 - xD + 4)\,y = \cos(\log x) + x \sin(\log x).$

$$\left[\text{Ans. } y = x[c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\cdot\log x)] + \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{x}{2}\sin(\log x)\right]$$

4. $(2x+1)^2 \dfrac{d^2y}{dx^2} - 2(2x+1)\dfrac{dy}{dx} - 12y = 6x.$

$$\left[\text{Ans. } y = c_1 (2x+1)^{-1} + c_2 (2x+1)^3 - \frac{3x}{8} + \frac{1}{16}\right].$$

5. $(x+2)^2 \dfrac{d^2y}{dx^2} - (x+2)\dfrac{dy}{dx} + y = 3x + 4.$

$$\left[\text{Ans } y = (x+2)\left[c_1 + c_2 \log(x+2) + \frac{3}{2}\log^2(x+2)\right] - 2.\right].$$

Examples : V—E

Solve :—

Ans.

1. $(D^2 - D - 6)\,y = e^x \cosh 2x.$ $\left[y = \left(c_1 + \dfrac{x}{10}\right)e^{3x} + c_2 e^{-2x} - \dfrac{e^{-x}}{8}\right]$

2. $r\dfrac{d^2y}{dr^2} + \dfrac{dy}{dr} - \dfrac{y}{r} = -ar^2.$ $\left[y = c_1 r - c_2 r^{-1} - \dfrac{ar^3}{8}\right].$

3. $(D^3 - D^2 + 3D + 5)\,y = e^x \cos 3x$

$$\left[y = c_1 e^{-x} + e^x(c_2 \cos 2x + c_3 \sin 2x) - \frac{e^x}{65}(3\sin 3x + 2\cos 3x)\right]$$

4. $(D^4 - m^4)\,y = \sin mx.$

$$\left[y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x}{4m^3}\cos mx.\right].$$

5. $(D^2 + 5D + 6)\,y = e^{-2x}\sin 2x + 4x^2 e^x$

$$\left[y = c_1 e^{-3x} + c_2 e^{-2x} - \frac{e^{-2x}}{10}(\cos 2x + 2\sin 2x) + \frac{e^x}{3}\left(x^2 - \frac{7}{6}x + \frac{37}{72}\right)\right].$$

6. $\dfrac{d^2v}{dr^2} + \dfrac{1}{r}\dfrac{dv}{dr} = A + B\log r.$

$$\left[v = (c_1 + c_2 \log r) + \frac{Ar^2}{4} + \frac{Br^2}{4}(\log r - 1)\right].$$

7. $(D^2 + 5D + 6)\,y = e^{-2x}(\sec^2 x)(1 + 2\tan x).$

$$[y = c_1 e^{-3x} + c_2 e^{-2x} + e^{-2x}\tan x].$$

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8. $(D^3 + 3D)\, y = \cosh 2x \sinh 3x$

$$\left[y = c_1 + c_2 \cos(\sqrt{3}\, x + c_3) + \frac{\cosh 5x}{200} + \frac{\cosh x}{8} \right].$$

9. $(D^4 - 2D^3 + D^2)\, y = x^3$.

$$\left[y = c_1 + c_2 x + (c_3 + c_4 x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2 \right].$$

10. $(D^2 - 2D + 4)^2 y = xe^x \cos(\sqrt{3}\, x + \alpha)$.

$$\left[\text{Ans. } y = e^x \left\{ (A + Bx)\cos\sqrt{3}\, x + (C + Dx)\sin(\sqrt{3}\, x) \right\} - \frac{x^2 e^x}{72} \left\{ x\cos(\sqrt{3}x + \alpha) - \sqrt{3}\sin(\sqrt{3}x + \alpha) \right\} \right]$$

11. $(D^2 + a^2)\, y = x^2 e^x$.

$$\left[\text{Ans. } y = c_1 \cos(ax + c_2) + \frac{e^x}{(a^2+1)^3} \left\{ (a^2+1)^2 x^2 - 4x(a^2+1) - 2(a^2-3) \right\} \right]$$

12. $(D^2 + 4)\, y = x \sin^2 x$.

$$\left[\text{Ans. } y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}\left(x - \frac{x\cos 2x}{4} - \frac{x^2 \sin 2x}{2} \right) \right]$$

13. $(D^2 - 4D + 4)\, y = 4(e^{2x} - \cos 2x)$.

$$\left[y = (c_1 + c_2 x)e^{2x} + 2x^2 e^{2x} + \frac{\sin 2x}{2} \right]$$

14. $(D^2 + 13D + 36)\, y = e^{-4x} + \sinh x$.

$$\left[\text{Ans. } y = c_1 e^{-4x} + c_2 e^{-9x} + \frac{x}{5} e^{-4x} - \frac{13\cosh x - 37\sinh x}{1200} \right]$$

15. $(D^3 - D^2 - D + 1)\, y = \cosh x \sin x$.

$$\left[\text{Ans. } y = c_1 e^{-x} + (c_2 + c_3 x)\, e^x + \frac{x^2 e^x}{8} + \frac{xe^{-x}}{8} + \frac{\sin x + \cos x}{4} \right]$$

16. $(D^3 + D)\, y = \cos x$. $\qquad \left[y = c_1 + c_2 \cos x + c_3 \sin x - \dfrac{x\cos x}{2} \right]$

17. $(D^3 + 1)\, y = \sin 3x - \cos^2 \dfrac{x}{2}$

$$\left[y = c_1 e^{-x} + c_2 e^{x/2} \cos\left(\frac{\sqrt{3}}{2}x + c_3 \right) + \frac{1}{730}(\sin 3x + 27\cos 3x) - \frac{1}{2} - \frac{1}{4}(\cos x - \sin x) \right]$$

18. $(D^3 - D^2 - 6D)\, y = x^2 + e^x \sin x$.

$$\left[\text{Ans. } y = c_1 + c_2 e^{3x} + c_3 e^{-2x} + \frac{x}{108}(6x^2 - 3x + 7) + \frac{e^x}{50}(\sin x + 7\cos x) \right]$$

19. $(D^2 - 3D + 2)\; y = e^x + x^2$.

$$\left[\text{Ans. } y = c_1 e^x + c_2 e^{2x} - xe^x + \frac{1}{2}\left(x^2 + 3x + \frac{7}{2} \right) \right]$$

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20. $(D^2 - 2D + 4)y = e^x \cos^2 x$.
$\left[\text{Ans. } y = c_1 e^x \cos(\sqrt{3}x + c_2) + \dfrac{e^x}{6} - \dfrac{e^x}{2} \cos 2x\right]$

21. $\dfrac{d^2x}{dt^2} + n^2 x = f \cos(nt + \alpha)$.
$\left[\text{Ans, } x = c_1 \cos nt + c_2 \sin nt + \dfrac{ft}{2n} \sin(nt + \alpha)\right]$

22. $(D^2 + 2D + 2)y = e^{-x} \sin x$.
$\left[\text{Ans. } y = c_1 e^{-x} \cos(x + c_2) - \dfrac{e^{-x}}{2}(x \cos x)\right]$

23. $(D^2 - 1)y = \cosh x \cos x$.
$\left[\text{Ans. } y = c_1 e^x + c_2 e^{-x} + \dfrac{1}{5}(2 \sinh x \sin x - \cosh x \cos x).\right]$.

24. $(D^2 - 3D + 2)y = x^2 e^x$.
$\left[\text{Ans. } y = (c_1 + c_2 x)e^x + c_3 e^{-2x} + \dfrac{x^2}{108} e^x (3x^2 - 4x + 4)\right]$.

25. $(D^2 - 1)y = x \sin x + (1 + x^2) e^x$.
$\left[\text{Ans. } y = c_1 e^x + c_2 e^{-x} - \dfrac{1}{2}(x \sin x + \cos x) + \dfrac{xe^x}{12}(2x^2 - 3x + 9)\right]$

26. $(D^2 + 2D + 1)y = x \cos x$.
$\left[\text{Ans. } y = (c_1 + c_2 x)e^{-x} + \dfrac{x - 1}{2} \sin x + \dfrac{1}{2} \cos x\right]$.

27. $(D^2 + 1)y = \tan x$. Hint : use method of variation of parameters
[Ans. $y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$].

28. $(D^2 + 2D + 2)y = \sinh x + x^2$.
$\left[\text{Ans. } y = e^{-x}(A \cos x + B \sin x) - \dfrac{1}{5}(2 \cosh x - 3 \sinh x) + (x - 1)^2\right]$.

29. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = xe^{-x} \sin x$
$\left[\text{Ans. } y = c_1 e^{-x} + c_2 e^{-2x} + \dfrac{1}{2} e^{-x} \{(x - 2) \cos x + (x - 1) \sin x\}\right]$.

30. $(D^2 + D)y = (1 + e^x)^{-1}$.
[Ans. $y = c_1 + c_2 e^{-x} - \log(1 + e^{-x}) - \frac{1}{2}e^{-x} \log(1 + e^x)$]

31. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 2 \log x + \dfrac{1}{x} + \dfrac{1}{x^2}$.
[Ans. $y = c_1 e^{2x} + c_2 e^{-x} - \log x$]

32. $(D^2 + 3D + 2)y = \sin(e^x)$ [Ans. $y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin(e^x)$]

33. $(D^4 - 6D^3 + 11D^2 - 6D)y = e^{-2t}$, $D \equiv \dfrac{d}{dt}$.
$\left[\text{Ans. } y = c_1 + c_2 e^t + c_3 e^{2t} + c_4 e^{3t} + \dfrac{e^{-2t}}{360}\right]$.

34. $(D^4 - D^3 - 7D^2 + 3D)\,y = 0.$

$$\left[\text{Ans. } y = c_1 + c_2 e^{3x} + e^{-x}\left(c_3 e^{\sqrt{2}\,x} + c_4 e^{-\sqrt{2}\,x}\right)\right].$$

35. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 5y = x^2 \log x.$

$$\left[\text{Ans. } y = x^2 \{c_1 \cos(\log x) + c_2 \sin(\log x)\} + x^2 \log x\right].$$

36. $4x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = x + \log x.$

$$\left[\text{Ans. } y = c_1 x + c_2 x^{-1/4} + \frac{x \log x}{5} + 3 - \log x\right].$$

37. $(x^3D^3 + 3x^2D^2 + xD)\,y = 24\,x^2.$

$$[\text{Ans. } y = 3x^2 + c_1 (\log x)^2 + c_2 \log x + c_3].$$

38. $(x^2D^2 - 3xD + 1)\,y = \dfrac{\sin(\log x.)}{x}.$

$$\left[\text{Ans. } y = x^2\left(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}\right) + \frac{1}{61x}(5 \sin \log x + 6 \cos \log x)\right]$$

39. $x \dfrac{d^3y}{dx^3} + 2 \dfrac{d^2y}{dx^2} = x^2$ $\left[\text{Ans. } y = c_1 + c_2 \log x + c_3 x + \dfrac{1}{48} x^4.\right]$

40. $x^2 \dfrac{d^2y}{dx^2} = 2y + \dfrac{1}{x}$ $\left[\text{Ans. } y = c_1 x^2 + \dfrac{c_2}{x} - \dfrac{11}{3x} \log x\right]$

41. $(D^2 + 6D + 9)\,y = \dfrac{1}{x^3} e^{-3x}.$ $\left[\text{Ans. } y = \left(\dfrac{1}{2x} + c_1 + c_2 x\right) e^{-3x}\right].$

42. $(D^2 - 2D + 2)\,y = e^x \tan x + 3x.$

$$\left\{\text{Ans } y = (c_1 \sin x + \{c_2 - \log(\sec x + \tan x)\} \cos x)e^x + \frac{3x}{2} + \frac{3}{2}\right\}.$$

43. $\left(\dfrac{d^2}{dx^2} - \dfrac{2}{x^2}\right)^2 y = 0$

$$\left[y = = c_1 x^4 + c_2 x^2 + c_3 x + c_4 \frac{1}{x}\right].$$

44. If $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 1$ and if $y = 0$ when $x = 0$ and when $x = \pm a$

prove that $y = a\left(\dfrac{\sinh x}{\sinh a} - \dfrac{x}{a}\right).$

45. $\dfrac{d^2y}{dx^2} + 6 \dfrac{dy}{dx} + 10\,y = 50x,$ with $y = 0, \dfrac{dy}{dx} = 1$ at $x = 0.$

$$[\text{Ans. } y = 5x - 3 + e^{-3x}(3 \cos x + 5 \sin x)].$$

46. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x \log x.$ $\left[\text{Ans. } y = x(c_1 + c_2 \log x) + \dfrac{x}{6}(\log x)^3\right]$

47. $(D^2 - 1)y = e^x + x^3$, with $y = \frac{dy}{dx} = 0$ at $x = 0$.

$$\left[\text{Ans. } y = \frac{11}{4}(e^x - e^{-x}) + \frac{1}{2}xe^x - x^3 - 6x\right].$$

48. Solve

$$\frac{d^4y}{dx^4} + 10\frac{d^2y}{dx^2} + 9y = 96 \sin 2x \cos x,$$

with initial conditions $x = 0, y = 0, \frac{dy}{dx} = -2, \frac{d^2y}{dx^2} = -8$ and $\frac{d^3y}{dx^3} = -18$.

$$\left[\text{Ans. } y = -\cos x + \cos 3x + x(\cos 3x - 3\cos x)\right].$$

49. Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + (1 + k^2)y = 0,$$

where k is a constant. Show that if $y = 0$ at $x = 0$, $\frac{dy}{dx} + 2y = 0$ at $x = 1$ and y is not identically zero, then k must be a root of the equation $\sin k + k \cos k = 0$.

50. Solve the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{x^3}{1 + x^2}$$

using the method of variation of parameter.

$$\left[\text{Ans. } y = Ax + \frac{B}{x} + \frac{x}{4}\log(1 + x^2) + \frac{x}{4} - \frac{1}{4x}\log(x^2 + 1)\right]$$

CHAPTER XIV

THE LAPLACE TRANSFORM

13. 1. Introduction :

Whenever a mathematical operator works on a function, the function is changed or transformed into another function. For example when the differential operator D $\left(= \frac{d}{dx}\right)$ works on $f(x) = \tan x$, it produces a new function

$$\phi(x) \equiv Df(x) = \sec^2 x.$$

The Laplace transform operator changes a function of one variable (usually called t) to a function of another variable (usually called s or p). The Laplace transform method changes an initial or a boundary value problem involving a differential equation to an algebraic equation.

It is proposed to give here only an introduction to the Laplace transform method.

13. 2. Definition of the Laplace Transform :

If $f(t)$ is a function of t, then the definite integral

$$\int_0^\infty e^{-st} f(t)\, dt$$

if it exists, will be a function of the parameter s, and is denoted by $\bar{f}(s)$.

There is one to one correspondence between $f(t)$ and $\bar{f}(s)$, and the relation transforms $f(t)$, a function of t into a new function $\bar{f}(s)$, which is a function of another variable s.

$f(t)$ is called the *object function*, which is defined for $t \geq 0$, $\bar{f}(s)$ is the resultant or *image function*, s is the parameter of the transform, which should be sufficiently large to make the integral convergent.

This relation between $f(t)$ and $\bar{f}(s)$, that is

$$\boxed{\bar{f}(s) = \int_0^\infty e^{-st}\,[f(t)]\,dt} \quad \ldots \quad \ldots \quad (1)$$

is written symbolically as

$$L\{\,f(t)\,\} = \bar{f}(s)$$

and $\bar{f}(s)$ is called the Laplace transform of $f(t)$.

It can be proved very easily from the fundamental definition (1) above, that the Laplace transformation is linear, that is

$$L\{\,AF_1(t) + BF_2(t)\,\} = AL\{\,F_1(t)\,\} + BL\{\,F_2(t)\,\}.$$

13·3. Table of Elementary Laplace Transforms :

With the help of the fundamental definition of Laplace transform, we can form a table of transforms for some simple functions.

(i) $f(t) = e^{at}$

By (1) above,

$$\bar{f}(s) = \int_0^\infty e^{-st}\,[\,e^{at}\,]\,dt = \int_0^\infty e^{-(s-a)\,t}\,dt$$

$$= \frac{1}{-(s-a)}\Big[e^{-(s-a)\,t}\Big]_0^\infty$$

With the restriction on s, that $s > a$, we have

$$\bar{f}(s) = -\frac{1}{(s-a)}\,[\,0-1\,] = \frac{1}{s-a}$$

Therefore the Laplace transform of e^{at} is $\dfrac{1}{s-a}$ and is written as

$$\boxed{L\{\,e^{at}\,\} = \frac{1}{s-a}} \quad \ldots \ldots \quad (2)$$

If in this we take $a = 0$, we have

$$\boxed{L\{1\} = \frac{1}{s}} \qquad \dots \qquad \dots \qquad \dots \qquad (3)$$

(ii) **f (t) = sin at**

As $\sin at = \frac{1}{2i}\left(e^{iat} - e^{-iat}\right)$, using the result (2) above, we have

$$\bar{f}(s) = \frac{1}{2i}\left[\frac{1}{s - ia} - \frac{1}{s + ia}\right] = \frac{a}{s^2 + a^2}$$

Therefore the Laplace transform of $\sin at$ is $\frac{a}{s^2 + a^2}$ and is written as

$$\boxed{L\{\sin at\} = \frac{a}{s^2 + a^2}} \qquad \dots \qquad \dots \qquad (4)$$

This result viewed from the fundamental definition (1) is

$$\int_0^\infty e^{-st} (\sin at)\, dt = \frac{a}{s^2 + a^2} \qquad \dots \qquad \dots \qquad (5)$$

If we differentiate both sides of the equation (5) w. r. t. the parameter a, we have

$$\int_0^\infty e^{-st} (t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

$$\therefore \quad L(at \cos at) = \frac{a(s^2 - a^2)}{(s^2 + a^2)^2} \qquad \dots \qquad \dots \qquad \dots \qquad (6)$$

$$\text{and } L(\sin at) = \frac{a}{s^2 + a^2} \qquad \dots \qquad - \qquad \dots \qquad (7)$$

Subtracting (6) from (7),

$$L(\sin at - at \cos at) = \frac{2a^3}{(s^2 + a^2)^2}.$$

Therefore

$$\boxed{L\left\{\frac{1}{2a^3}(\sin at - at \cos at)\right\} = \frac{1}{(s^2 + a^2)^2}} \qquad \dots \qquad (8)$$

(iii) $f(t) = \cos at$.

As $\cos at = \frac{1}{2}\left[e^{iat} + e^{-iat}\right]$, using the result (2), we have

$$f(s) = \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{s}{s^2+a^2}$$

Therefore the Laplace transform of $\cos at$ is $\frac{s}{s^2+a^2}$ and written as

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}} \quad \ldots \quad \ldots \quad (9)$$

The result (9) means

$$\int_0^\infty e^{-st}(\cos at)\,dt = \frac{s}{s^2+a^2} \quad \ldots \quad \ldots \quad (10)$$

If we differentiate (10) w. r. t. the parameter a, we have one more transform result. Thus

$$\int_0^\infty e^{-st}(t \sin at)\,dt = \frac{2as}{(s^2+a^2)^2}$$

Therefore

$$\boxed{L\left\{\frac{t}{2a}\sin at\right\} = \frac{s}{(s^2+a^2)^2}} \quad \ldots \quad \ldots \quad (11)$$

(iv) **$f(t) = \sinh at$** and (v) **$f(t) = \cosh at$.**

As $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

and $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$,

we have by using (2), in a similar manner as above,

$$\boxed{L(\sinh at) = \frac{a}{s^2-a^2}} \quad \ldots \quad \ldots \quad (12)$$

and

$$\boxed{L(\cosh at) = \frac{s}{s^2 - a^2}} \quad \ldots \quad \ldots \quad (13)$$

(v) $\mathbf{f(t) = t^n}$

$$L(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

put $st = x$

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$$\therefore L(t^n) = \int_0^\infty e^{-x} \frac{x^n}{s^n} \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

If n is a positive integer $\Gamma(n+1) = n!$

$$\therefore Lt^n = \frac{n!}{s^{n+1}}, \text{ if } n \text{ is } +\text{ve integer}$$

Thus

$$\boxed{\begin{aligned} L(t^n) &= \frac{\Gamma(n+1)}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}} \quad \text{for } n + \text{ve integer} \end{aligned}} \quad \ldots \quad \ldots \quad (14)$$

We shall now collect all the above results of the Laplace tranforms of elementary functions in the form of a table for ready reference.

Table of Laplace Transforms

$f(t)$	$\bar{f}(s)$
1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a},\ s>a$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$\sinh at$	$\dfrac{a}{s^2-a^2}$
$\cosh at$	$\dfrac{s}{s^2-a^2}$
$\dfrac{t}{2a}\sin at$	$\dfrac{s}{(s^2+a^2)^2}$
$\dfrac{1}{2a^3}(\sin at - at\cos at)$	$\dfrac{1}{(s^2+a^2)^2}$
t^n	$\dfrac{\Gamma(n+1)}{s^{n+1}}$

For our elementary work, this small table will be sufficient.

13.4. Theorems on Important Properties of Laplace Transforms :

Following theorems are useful to obtain the Laplace Transforms of functions from those of elementary functions mentioned in art. 3 above.

(A) First Shifting Theorem :

If $L\{f(t)\}=\bar{f}(s)$, then $L\{e^{-at}f(t)\}=\bar{f}(s+a)$

Proof :

$$L\{e^{-at}f(t)\}=\int_0^\infty e^{-st}\{e^{-at}f(t)\}\,dt$$

$$=\int_0^\infty e^{-(s+a)t}f(t)\,dt$$

$$=\int_0^\infty e^{-pt}f(t)\,dt \;[\text{where } p=s+a]$$

$$=\bar{f}(p)$$

$$=\bar{f}(s+a)$$

Therefore

$$\boxed{L\{e^{-at}f(t)\}=\bar{f}(s+a)} \qquad \text{------} \qquad (15)$$

We note that if the object function is multiplied by e^{-at} then in the Laplace trnsform of the object function, s is replaced by $(s+a)$ and so the name theorem on substitution.

Example : Find the Laplace tranceform of (i) $e^{-bt}\cos at$ (ii) t^2e^{3t}

(i) From the result $L(\cos at)=\frac{s}{s^2+a^2}$ and (15).

we get

$$L\left\{e^{-bt}\cos at\right\}=\frac{s+b}{(s+b)^2+a^2}.$$

(ii) $L(t^2)=\frac{2!}{s^3}$ $\therefore$ By (15), $L(t^2e^{3t})=\frac{2!}{(s-3)^3}$

(B) Second Shifting Theorem :

If $L\{f(t)\}=\bar{f}(s)$ and $F(t)=\begin{cases} f(t-a) & t>a \\ 0 & t<a\end{cases}$

then

$$L\{F(t)\}=e^{-as}\bar{f}(s)$$

Proof :—

$$L\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt = \int_0^{a} e^{-st} F(t)\, dt + \int_a^{\infty} e^{-st} F(t)\, dt$$

$$= \int_0^{a} e^{-st} (0)\, dt + \int_a^{\infty} e^{-st} f(t-a)\, dt$$

$$= \int_a^{\infty} e^{-st} f(t-a)\, dt$$

$$= \int_0^{\infty} e^{-s(u+a)} f(u)\, du, \quad [u = t - a)$$

$$= e^{-as} \int_0^{\infty} e^{-su} f(u)\, du$$

$$= e^{-as} \bar{f}(s)$$

Hence

$$\boxed{L\{F(t)\} = e^{-as} \bar{f}(s) \quad \text{where } F(t) = \begin{cases} f(t-a), & t > a \\ 0 & t \leq a \end{cases}} \quad \ldots (16)$$

Example : Find $L\{F(t)\}$ for

$$F(t) = \begin{cases} (t-1)^3 & t > 1 \\ 0 & 0 < t < 1 \end{cases}$$

Here $f(t) = t^3$, hence $\bar{f}(s) = \dfrac{3!}{s^4}$

$\therefore$ By above theorem as $a = 1$

$$L\{F(t)\} = \frac{3!\, e^{-s}}{s^4}$$

(C) If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{f}(s)$$

Now $\bar{f}(s) = \displaystyle\int_0^{\infty} e^{-st} f(t)\, dt$

By differentiating under the integral sign

$$\frac{d}{ds}\bar{f}(s)=\frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt=\int_0^\infty \frac{\partial}{\partial s}\{e^{-st}f(t)\,dt\}$$

$$=\int_0^\infty -t\,e^{-st}f(t)\,dt$$

$$=-\int_0^\infty e^{-st}\{t f(t)\}\,dt.$$

Hence by definition of Laplace transform, we have

$$L\{t f(t)\}=-\frac{d}{ds}\bar{f}(s)$$

Hence by using mathematical induction, we can show that

$$\boxed{L\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}\bar{f}(s)}\qquad \ldots \qquad (17)$$

Cor. The result

$$L\{tf(t)\}=-\frac{d}{ds}\bar{f}(s)=-\bar{f}'(s)$$

can be interpreted that the *differentiation of the transform of a function corresponds to the multiplication of the function by* $-t$. This result is useful in finding inverse transform as is illustrated in example (iii) of art. (111).

Example : Find $L\{f(t)\}$ for

(i) $\frac{t}{2a}\sinh at$ (ii) $t^2\cos at$

(i) $f(t)=\frac{\sinh at}{2a}$ $\bar{f}(s)=\frac{1}{2}\cdot\frac{1}{s^2-a^2}$

$$\therefore\quad L\left\{t\,\frac{1}{2a}\sinh at\right\}=(-1)\frac{d}{ds}\left\{\frac{1}{2}\cdot\frac{1}{s^2-a^2}\right\}\text{[by (17)]}$$

$$=(-1)\cdot\frac{1}{2}\cdot\frac{-2s}{(s^2-a^2)^2}$$

$$=\frac{s}{(s^2-a^2)^2}$$

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(ii) $f(t) = \cos at$, $\overline{f}(s) = \dfrac{s}{s^2+a^2}$

$$\therefore \; L\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\}$$

$$= \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}.$$

(D) If $Lf(t) = \bar{f}(s)$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds, \text{ provided } \lim_{t\to+0} \frac{f(t)}{t} \text{ exists}$$

$$\int_s^\infty \bar{f}(s)\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt\right] ds$$

$$= \int_0^\infty f(t) \left[\int_0^\infty e^{-st}\, ds\right] dt$$

(by changing the order of integration)

$$= \int_0^\infty \frac{f(t)}{t} e^{-st}\, dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

Hence

$$\boxed{L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds} \qquad \ldots \qquad \ldots \qquad (18)$$

Example : Find the Laplace transform of $\sin at$ and hence show that

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

Here take $f(t) = \sin at$

$$\therefore \; \overline{f}(s) = \frac{s}{s^2+a^2}$$

Hence by result (18), we get

$$L\left(\frac{\sin at}{t}\right)=\int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$=\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$=\frac{\pi}{2}-\tan^{-1}\frac{s}{a}$$

$$=\cot^{-1}\left(\frac{s}{a}\right)$$

when $a=0$

$$L\left(\frac{\sin t}{t}\right)=\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt=\cot^{-1}(s).$$

when $s=0$, $$\int_0^\infty \frac{\sin t}{t}\,dt=\cot^{-1}(0)=\frac{\pi}{2}$$

(E) If $L\{f(t)\}=f(s)$, then

$$L\{f(at)\}=\frac{1}{a}f\left(\frac{s}{a}\right)$$

$$Lf(at)=\int_0^\infty e^{-st}f(at)\,dt$$

$$=\int_0^\infty e^{-s(u/a)}f(u)\,\frac{du}{a},\ [u=at]$$

$$=\frac{1}{a}\int_0^\infty e^{-(s/a)u}f(u)\,du$$

$$=\frac{1}{a}f(s/a),\ \text{by definition}$$

Thus

$$\boxed{L\{f(at)\}=\frac{1}{a}f\left(\frac{s}{a}\right)} \quad \ldots \quad (19)$$

Example : If $L[f(t)] = \dfrac{8+12s-2s^2}{(s^2+4)^2}$, find $L\{f(2t)\}$.

By result (19),

$$L\{f(2t)\} = \frac{1}{2}\left\{\frac{8+12\left(\frac{s}{2}\right)-2\left(\frac{s}{2}\right)^2}{[(s/_2)^2+4]^2}\right\} = \frac{4(16+12s-s^2)}{(s^2+16)^2}$$

(F) The Convolution Theorem

This theorem is useful to find a function F (t) whose transform $\bar{F}(s)$ is not the transform of a known function, by expressing $\bar{F}(s)$ as the product of two functions each of which is the transform of a known function. i. e.

$$\bar{F}(s) = \bar{f}_1(s)\,\bar{f}_2(s)$$

where $\bar{f}_1(s)$ and $\bar{f}_2(s)$ are transforms of known functions

$f_1(t)$ and $f_2(t)$

The theorem states that

$$L\left\{\int_0^t f_1(t-u)\,f_2(u)\,du\right\} = \bar{f}_1(s)\,\bar{f}_2(s)$$

Proof :-

By definition of Laplace transform, we have

$$\bar{f}_1(s)\,\bar{f}_2(s) = \left[\int_0^\infty e^{-sv} f_1(v)\,dv\right]\left[\int_0^\infty e^{-su} f_2(u)\,du\right]$$

$$= \int_0^\infty\int_0^\infty e^{-s(u+v)} f_1(v)\,f_2(u)\,dv\,du$$

$$= \int_0^\infty f_2(u)\,du\left[\int_0^\infty e^{-s(u+v)} f_1(v)\,dv\right]$$

$$= \int_0^\infty f_2(u)\,du\left[\int_u^\infty e^{-st} f_1(t-u)\,dt\right]$$

[where $u + v = t$]

Changing the order of double integration,

$$= \int_0^\infty \left[\int_0^t e^{-st} f_1(t-u) f_2(u)\, du \right] dt$$

$$= \int_0^\infty e^{-st} \left[\int_0^t f_1(t-u) f_2(u)\, du \right] dt$$

$$= L\left[\int_0^t f_1(t-u) f_2(u)\, du \right], \text{ by definition.}$$

Hence

$$\boxed{\begin{aligned} L\left[\int_0^t f_1(t-u) f_2(u)\, du \right] &= \bar{f}_1(s)\, \bar{f}_2(s) \\ &= L\left[\int_0^t f_1(u) f_2(t-u)\, du \right] \end{aligned}} \quad \ldots (20)$$

This theorem is useful to find inverse transformation.

Example ; Verify the convolution theorem for the pair of functions

$$f_1(t) = t,\ f_2(t) = e^{at}$$

Here

$$\overline{f_1}(s) = \frac{1}{s^2} \text{ and } \overline{f_2}(s) = \frac{1}{s-a}$$

$$\therefore \overline{f_1}(s) . \overline{f_2}(s) = \frac{1}{s^2(s-a)} \quad \ldots \quad \ldots \quad \ldots$$

Now

$$\int_0^t f_1(u) f_2(t-u)\, du = \int_0^t u\, e^{a(t-u)}\, du$$

$$= \left[-\frac{u}{a} e^{a(t-u)} - \frac{1}{a^2} e^{a(t-u)} \right]_0^t$$

$$= \frac{1}{a^2} \left[e^{at} - at - 1 \right]$$

$$\therefore\; L\left\{\int_0^t f_1(u)\, f_2(t-u)\, du\right\}$$

$$= L\left\{\frac{1}{a^2}\left(e^{at} - at - 1\right)\right\}$$

$$= \frac{1}{a^2}\left[\frac{1}{s-a} - \frac{a}{s^2} - \frac{1}{s}\right] = \frac{1}{s^2(s-a)}$$

$$= \bar{f}_1(s)\, \bar{f}_2(s) \;[\text{ from (i) }]$$

(G) Laplace Transform of an Integral

By definition,

$$L\left\{\int_0^t f(u)\, du\right\} = \int_0^\infty e^{-st}\left[\int_0^t f(u)\, du\right] dt \quad \ldots \quad \ldots \quad \text{(i)}$$

Since

$$\frac{d}{dt}\left[\int_0^t f(u)\, du\right] = f(t),$$

we get, by integrating by parts, the result (i) as

$$L\left\{\int_0^t f(u)\, du\right\} = \left[-\frac{1}{s}\, e^{-st}\int_0^t f(u)\, du\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\, dt$$

$$= \frac{1}{s}\, \bar{f}(s).$$

Thus

$$\boxed{L\left[\int_0^t f(u)\, du\right] = \frac{1}{s}\, \bar{f}(s)} \quad \ldots \quad (21)$$

i. e. *If a function is integrated over* $(0, t)$ *the transform of the integral is obtained by dividing the transform of the function by s.*

Example : Verify directly

$$L\left\{\int_0^t u^2 e^{-u}\,du\right\} = \frac{1}{s}\,L\left\{t^2 e^{-t}\right\}$$

Now $$\int_0^t u^2 e^{-u}\,du = \Big[-(u^2+2u+2)\,e^{-u}\Big]_0^t$$

$$= 2-(t^2+2t+2)\,e^{-t}$$

$$\therefore\quad L\left\{\int_0^t u^2 e^{-u}\,du\right\} = L\left\{2-(t^2+2t+2)\,e^{-t}\right\} \quad \ldots \quad \ldots \text{(i)}$$

Now by result

$$L(t^2 e^{-t}) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+1}\right) = \frac{2}{(s+1)^3}$$

$$L(2te^{-t}) = 2.(-1)\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{2}{(s+1)^2}$$

$\therefore$ From (i), we have

$$L\left\{\int_0^t u^2 e^{-u}\,du\right\} = \frac{2}{s} - \left[\frac{2}{(s+1)^3}+\frac{2}{(s+1)^2}+\frac{2}{s+1}\right]$$

$$= \frac{2}{s\,(s+1)^3} = \frac{1}{s}\,L\left\{t^2 e^{-t}\right\}.$$

(H) Laplace Transform of Derivatives

We can express the transform of any derivative of the function $f(t)$ in terms of the transform of the function itself and in terms of the values of the lower order derivatives of the function at $t = 0$ (i. e. values approached by the derivatives as $t \to 0$ from positive values).

If $L[f(t)] = \bar{f}(s)$ and $f(t)$ is continuous and is of exponential order s_0, $\left[\text{i. e. } \lim\limits_{m\to\infty} e^{-ms} f(m) = 0, \text{ for } s > s_0\right]$, then

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

where $f(0)$ is the value of $f(t)$ at $t = 0$.

Proof :—Integrating by parts, we get

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \lim_{m \to \infty} \int_0^m e^{-st} f'(t)\, dt$$

$$= \lim_{m \to \infty} \left\{ \Big[e^{-st} f(t) \Big]_0^m + s\int_0^m e^{-st} f(t)\, dt \right\}$$

$$= \lim_{m \to \infty} \left\{ e^{-sm} f(m) - f(0) \right\} + s\int_0^m e^{-st} f(t)\, dt$$

$$= s\int_0^\infty e^{-st} f(t)\, dt - f(0)$$

[as $f(t)$ is of exponential order].

Hence

$$\boxed{L\{f'(t)\} = s\bar{f}(s) - f(0)} \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Corollary :—

If $L\{f(t)\} = \bar{f}(s)$, then $L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$

Let $F(t) = f'(t)$, then

$$L\{f''(t)\} = L\{F'(t)$$

$$= sL\{F(t)\} - F(0) \quad \text{[by result (22)]}$$

$$= sL\{f'(t)\} - f'(0)$$

$$= s[s\bar{f}(s) - f(0)] - f'(0) \quad \text{[by result(22)]}$$

$$= s^2\bar{f}(s) - sf(0) - f'(0)$$

By using mathematical induction, we can show that

$$\boxed{\begin{aligned} L\left\{f^{(n)}(t)\right\} &= s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots \\ &\quad - sf^{(n-2)}(0) - f^{(n-1)}(0) \end{aligned}} \quad \ldots \quad (23)$$

Example : Obtain the Laplace transform of $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 5y$ being given that $y(0) = 2$ and $y'(0) = -4$.

Now by result (21) and using $\bar{y}(s) = L\{y(t)\}$

$$L\left\{\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 5y\right\} = L\left(\frac{d^2y}{dt^2}\right) - 3L\left(\frac{dy}{dt}\right) + 5L(y)$$

$$= [s^2\bar{y}(s) - s\,y(0) - y'(0)]$$

$$-3[s\bar{y}(s) - y(0)] + 5\bar{y}(s)$$

$$= [s^2\bar{y}(s) - 2s + 4] - 3[s\bar{y}(s) - 2] + 5\bar{y}(s)$$

$$[\text{as } y(0) = 2,\ y'(0) = -4]$$

$$= (s^2 - 3s + 5)\,\bar{y}(s) - 2s + 10$$

Solved Problems :—

Example 1. Find the Laplace transform of each of the following functions

(i) $\cos t \cdot \cos 2t$ (ii) $t^2 - 3t + 5$

(iii) $t^2 \sin at$ (iv) $e^{4t}\cosh 5t$

(v) $t\,e^t f(t)$.

(i) $L\{\cos t \cdot \cos 2t\} = L\{\tfrac{1}{2}(\cos 3t + \cos t)\}$

$$= \tfrac{1}{2}\{L(\cos 3t) + L(\cos t)\}$$

$$= \tfrac{1}{2}\left\{\frac{s}{s^2+(3)^2} + \frac{s}{s^2+(1)^2}\right\} \quad \text{from the table}$$

$$= \frac{s(s^2+5)}{(s^2+1)[s^2+9)}$$

(ii) From table of transforms

$$L\left(t^{n-1}\right) = \frac{(n-1)!}{s^n}$$

Hence, we have

$$L(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}, \quad n=3$$

$$L(t) = \frac{1}{s^2}, \quad n=2$$

$$L(1) = \frac{1}{s}, \quad n=1$$

$$\therefore L(t^2 - 3t + 5) = L(t^2) - 3L(t) + 5L(1)$$

$$= \frac{2}{s^3} - \frac{3}{s^2} + \frac{5}{s} = \frac{5s^2 - 3s + 2}{s^3}$$

Laplace Transform

(iii) Using theorem (C) of art. 4, we have

$$L\{t^2 \sin at\} = (-1^2)\frac{d^2}{ds^2} \cdot \left\{\frac{a}{s^2+a^2}\right\}$$

$$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

(iv) Using the theorem (A) of art. 4, and result (13), we get

$$L\{e^{4t}\cosh 5t\} = \frac{s-4}{(s-4)^2-5^2}$$

$$= \frac{s-4}{s^2-8s-9}$$

e^{4t} $\frac{s-4}{(s-4)^2-5^2}$

(v) Let $L\{f(t)\} = \bar{f}(s)$, then using theorem (A), we get

$$L\{e^t f(t)\} = \bar{f}(s-1)$$

Now using theorem (C), we have

$$L\{t[e^t f(t)]\} = (-1)\frac{d}{ds}\bar{f}(s-1)$$

$$= -\bar{f}'(s-1)$$

Example 2. Find $L\{f(t)\}$, if $f(t) = \begin{cases} \cos(t-\alpha), & t>\alpha \\ 0, & t<\alpha \end{cases}$

By definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\alpha e^{-st}(0)\,dt + \int_\alpha^\infty e^{-st}\cos(t-\alpha)\,dt$$

$$= \int_0^\infty e^{-s(u+\alpha)}\cos u\,du \quad [\text{ where } u=t-\alpha]$$

$$= e^{-\alpha s}\int_0^\infty e^{-su}\cos u\,du$$

$$= e^{-\alpha s} L\{\cos u\} = e^{-s\alpha}\frac{s}{s^2+1}$$

Aliter :—Since $L\{\cos t\} = \frac{s}{s^2+1}$, we get, by using theorem (B) of art 4,

$$L\{f(t)\} = e^{-\alpha s}\frac{s}{s^2+1} \quad [\text{here } a=\alpha].$$

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Example 3 : Find Laplace transform of

(i) $t^{5/2}$ (ii) $e^{-3t}\, t^{-1/2}$ (iii) $erf\sqrt{t}$

By result (14)

(i) $L\{t^{5/2}\} = \frac{\Gamma(7/2)}{s^{7/2}} = \frac{15}{8}\,\frac{\Gamma(1/2)}{s^{7/2}} = \frac{15}{8}\sqrt{\frac{\pi}{s^7}}$

(ii) $L\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$

$\therefore \; L\{e^{-3t}t^{-1/2}\} = \sqrt{\frac{\pi}{s+3}}$ by result (15).

(iii) By definition of error function

$$erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{\pi}}\int_0^{t} u^{-1/2}\,e^{-u}\,du \qquad [x^2 = u]$$

$$L\{erf\sqrt{t}\} = \frac{1}{\sqrt{\pi}}\,L\left\{\int_0^t u^{-1/2}\,e^{-u}\,du\right\}$$

$$= \frac{1}{\sqrt{\pi}}\;\frac{1}{s}\,L\{u^{-1/2}e^{-u}\} \qquad \text{by result (21)}$$

$$= \frac{1}{\sqrt{\pi}}\;\frac{1}{s}\cdot\frac{\Gamma(1/2)}{(s-1)^{1/2}} = \frac{1}{s\sqrt{s-1}}$$

Example 4 :—Given $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$, show that

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$$

Let $f(t) = 2\sqrt{\frac{t}{\pi}}$

$\therefore\; f'(t) = \frac{2}{\sqrt{\pi}}\;\frac{1}{2}\;t^{-1/2} = \frac{1}{\sqrt{\pi t}}$

$\therefore\; Lf'(t) = L\frac{1}{\sqrt{\pi}} = s\,\bar{f}(s) - f(0)$ [by result (22)]

$$= sL\left\{2\sqrt{\frac{t}{\pi}}\right\}$$

$$= s \cdot \frac{1}{s^{3/2}} = \frac{1}{\sqrt{s}}$$

Example 5 : Evaluate $\int_0^{\infty} t e^{-3t} \sin t \, dt$

$$\int_0^{\infty} t e^{-3t} \sin t \, dt = \int_0^{\infty} e^{-st} (t \sin t) \, dt \qquad (\text{where } s=3)$$

$$= L\{t \sin t\} \quad \text{by definition}$$

$$= (-1) \frac{d}{ds} \left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{2s}{(s^2+1)^2}$$

$$= \frac{6}{100} = \frac{3}{50} \quad [\text{replacing } s=3]$$

Examples XIII-A

1. Obtain the Laplace transform of each of the following functions :—

(i) $t^2 - e^{-2t} + e^{t}$ (ii) $\frac{1}{4}t - \frac{1}{3} \sin t + \frac{1}{24} \sin 2t$

(iii) $(t^2+1)^2$ (iv) $(\sin 2t - \cos 2t)^2$

(v) $e^{at}\{2 \cos bt - 3 \sin bt\}$ (vi) $t^n e^{-at}$

(vii) $(t+1)^2 e^t$ (viii) $a \cos^2 2bt$

(ix) $\cosh^2 4t$ (x) $t(2 \sin 3t - 3 \cos 3t)$

(xi) $t^3 \cos kt$ (xii) $t e^{3t} \sin 2t$

(xiii) $\cos at \sinh at$ (xiv) $\sin^3 t$ (xv) $\sin(\omega t + \alpha)$

(xvi) $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$ (xvii) $e^{-t} \sin^2 t$

(xviii) $2e^t \sin 4t \cos 2t$ (xix) $e^{4t} t^{3/2}$

Ans. (i) $\frac{3s^3+2s^2+2s-4}{s^3(s+2)(s-1)}$ (ii) $\frac{1}{s^2(s^2+1)(s^2+4)}$

(iii) $\frac{s^4+4s^2+24}{s^5}$ (iv) $\frac{1}{s} - \frac{4}{s^2+16}$

(v) $\frac{2s-2a-3b}{(s-a)^2+b^2}$ (vi) $\frac{n!}{(s+a)^{n+1}}$

(vii) $\dfrac{s^2+1}{(s-1)^3}$ (viii) $\dfrac{a}{2s}+\dfrac{as}{2s^2+32b^2}$

(ix) $\dfrac{1}{2}\left[\dfrac{1}{s}+\dfrac{s}{s^2-64}\right]$ (x) $\dfrac{3(9-4s+s^2)}{(s^2+9)^2}$

(xi) $\dfrac{2s(s^2-3k^2)}{(s^2+k^2)^3}$ (xii) $\dfrac{4(3-s)}{(s^2-6s+13)^2}$

(xiii) $\dfrac{a(s^2-2a^2)}{s^4+4a^4}$ (xiv) $\dfrac{6}{(s^2+1)(s^2+9)}$

(xv) $\dfrac{as+b\omega}{s^2+\omega}$, where $\tan\alpha=\dfrac{a}{b}$

(xvi) $s\cot^{-1}s-1$ (xvii) $\dfrac{2}{(s+1)(s^2+2s+1)}$ $(s^2+1)^2$

(xviii) $\dfrac{6}{s^2-2s+37}+\dfrac{2}{s^2-2s+5}$ (xix) $\dfrac{3}{4}\sqrt{\dfrac{\pi}{s-4}}$.

2. By using the fundamental definition, find the Laplace transform of $f(t)$, where

(i) $f(t)=\begin{cases} a, & 0<t<b \\ 0, & t>b \end{cases}$ ans. $\dfrac{a(1-e^{-bs})}{s}$

(ii) $f(t)=\begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$ ans. $\dfrac{1}{s^2}+\left(\dfrac{1}{s}-\dfrac{1}{s^2}\right)e^{-4s}$

(iii) $f(t)=\begin{cases} 0, & 0<t<1 \\ t, & 1<t<2 \\ 0, & t>2 \end{cases}$ ans. $\left(\dfrac{1}{s^2}+\dfrac{1}{s}\right)e^{-s}-\left(\dfrac{1}{s^2}+\dfrac{2}{s}\right)e^{-2s}$

(iv) $f(t)=\begin{cases} \sin 2t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$ ans. $\dfrac{2(1-e^{-\pi s})}{s^2+4}$

(v) $f(t)=\begin{cases} (t-1)^2, & t>1 \\ 0, & 0<t<1 \end{cases}$ ans. $\dfrac{2e^{-s}}{s^3}$

3. If $L\{f(t)\}=\dfrac{1}{s}e^{-1/s}$, find $L\{e^{-t}f(3t)\}$

$$\left[\text{Ans. } \frac{e^{-3/(s+1)}}{S+1}\right]$$

4. If $L\{erf\sqrt{t}\}=\dfrac{1}{s(s+1)}$ find $L\{t\,erf(2\sqrt{t})\}$

$$\left[\text{Ans. } \frac{3s+8}{s^2(s+4)^{3/2}}\right]$$

5. Given $f(t)=t+1,\quad 0<t<2$
$\qquad\qquad =3 \qquad\quad t>2$

find $L\{f(t)\}$ and $L\{f'(t)\}$.

$$\left[\text{Ans. } \frac{1}{s}+\frac{1}{s^2}(1-e^{-2s}),\ \frac{1}{s}(1-e^{-2s})\right]$$

6. Given $L\, J_0(t)=\dfrac{1}{\sqrt{1+s^2}}$, show that

(i) $L\{t J_0(at)\}=\dfrac{s}{(s^2+a^2)^{3/2}}$

(ii) $L\{e^{-at} J_0(at)\}=\dfrac{1}{\sqrt{s^2+2as+2a^2}}$

(iii) $\displaystyle\int_0^{\infty} J_0(t)\,dt=1$

(iv) $\displaystyle\int_0^{\infty} t\,e^{-3t}\, J_0(4t)=\frac{3}{125}$

7. Find the Laplace transforms of :—

(i) $\dfrac{1}{t}(1-\cos at)$ (ii) $\dfrac{1}{t}\left(e^{at}-e^{bt}\right)$

(iii) $\dfrac{1}{t}(\cos at-\cos bt)$ (iv) $\dfrac{\sinh t}{t}$ (v) $t^{-1}e^{-t}\sin t$

(vi) $\displaystyle\int_0^t e^t\,\frac{\sin t}{t}\,dt$ (vii) $\displaystyle\int_0^t x\cosh x\,dx$

(viii) $\cosh t\displaystyle\int_0^t e^x\cosh x\,dx$ (ix) $\dfrac{\sin^2 t}{t}$

$\Big[$Ans. (i) $\dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2}\right)$ (ii) $\log\left(\dfrac{s-b}{s-a}\right)$

(iii) $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$ (iv) $\dfrac{1}{2}\log\left(\dfrac{s+1}{s-1}\right)$

(v) $\cot^{-1}(s+1)$. (vi) $\dfrac{1}{s}\cot^{-1}(s-1)$

(vii) $\dfrac{s^2+1}{s(s^2-1)^2}$

(viii) $\dfrac{1}{2}\left[\dfrac{s-2}{(s-1)^2(s-3)}-\dfrac{s}{(s+1)^2(s-1)}\right]$

(ix) $\dfrac{1}{2}\log\left(\dfrac{s^2+2}{s^2}\right)$

8. If $L f(t) = \dfrac{s^2 - s + 1}{(2s+1)^2 (s-1)}$, find $L(2t)$.

$$\left[\text{Ans. } \frac{s^2-2s+4}{4(s+1)^2(s-2)}\right]$$

9. Evalute

(i) $\displaystyle\int_0^\infty t^2 e^{-t} \sin t \, dt$ (ii) $\displaystyle\int_0^\infty e^{-2t} \sin^2 t \, dt$

(iii) $\displaystyle\int_0^\infty t e^{-3t} \sin t \, dt$ (iv) $\displaystyle\int_0^\infty e^{-2t} \frac{\sinh t}{t} dt$

(v) $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$

$$\left[\text{Ans. (i) } 0 \quad \text{(ii) } \frac{6}{65} \quad \text{(iii) } \frac{3}{50} \quad \text{(iv) } \frac{1}{2}\log 3 \quad \text{(v) } \log\frac{2}{3}\right]$$

10. The function $J_0(t)$ satisfies the differential equation

$$t J_0''(t) + J_0'(t) + t J_0(t) = 0$$

and if $J_0(0) = 1$ and $J_0'(0) = 0$, show that

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

given that

$$\int_0^\infty J_0(t)\, dt = 1.$$

13.5 Inverse Laplace Transform

Given the object function $f(t)$, we are now in a position to find the Laplace transform $\bar{f}(s)$. We shall now consider the inverse problem, that is given $\bar{f}(s)$, to find the object function $f(t)$ of which $\bar{f}(s)$ is the Laplace transform.

If $L\{f(t)\} = \bar{f}(s)$, then $f(t)$ is called the inverse Laplace transform of $\bar{f}(s)$ and this inverse relation is denoted by

$$\boxed{L^{-1}\left\{\bar{f}(s)\right\} = f(t)} \quad \ldots \quad \ldots \quad (22)$$

Example : Find (i) $L^{-1}\left\{\frac{1}{s+4}\right\}$ (ii) $L^{-1}\left\{\frac{2s+6}{s^2+4}\right\}$

(i) From table of transforms

$$L^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$$

$$\text{or} \quad f(t) = e^{-4t}$$

(ii) $$\frac{2s+6}{s^2+4} = 2\,\frac{s}{s^2+4} + 3\,\frac{2}{s^2+4} \quad \text{(note the step)}$$

From the table,

$$L(\cos 2t) = \frac{s}{s^2+4}, \quad L(\sin 2t) = \frac{2}{s^2+4}$$

$$\therefore \quad L^{-1}\left\{\frac{2s+6}{s^2+4}\right\} = 2L^{-1}\left\{\frac{s}{s^2+4}\right\} + 3L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 2\cos 2t + 3\sin 2t$$

Following are some of the methods to find inverse Laplace transform by using the known Laplace transforms of elementary functions.

(I) Shifting Theorem :

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then

$$\boxed{L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t)} \quad \ldots \quad (23)$$

Example : Find $f(t)$, if $\bar{f}(s) = \dfrac{s+7}{s^2+2s+5}$.

We complete a square with the first two terms in the denominator, thus

$$s^2 + 2s + 5 = (s+1)^2 + (2)^2$$

Hence

$$\bar{f}(s) = \frac{s+7}{s^2+2s+5} = \frac{(s+1)}{(s+1)^2+2^2} + 3\,\frac{2}{(s+1)^2+(2)^2}$$

In splitting $\bar{f}(s)$, note has been taken of Laplace transforms of $\sin at$, $\cos at$ and the shifting theorem.

From the table

$$L\{\cos 2t\} = \frac{s}{s^2+(2)^2}, \quad L\{\sin 2t\} = \frac{2}{s^2+(2)^2}$$

Hence by the shifting theorem we have

$$L^{-1}\left\{\frac{s+1}{(s+1)^2+(2)^2}\right\} = e^{-t}\cos 2t$$

$$L^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\} = e^{-t}\sin 2t$$

$$\therefore\ f(t) = L^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}$$

$$= L^{-1}\left\{\frac{s+1}{(s+1)^2+(2)^2}\right\} + 3\,L^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\}$$

$$= e^{-t}\cos 2t + 3\,e^{-t}\sin 2t..$$

(II) Partial fractions Methods :

Generally in many problems $\bar{f}(s)$ is a rational fraction $\dfrac{\bar{F}(s)}{\bar{G}(s)}$ with degree of $\bar{F}(s)$ less than that of $\bar{G}(s)$ and this fraction can be expressed as sum of partial fractions of the type $\dfrac{A}{(as+b)^r}, \dfrac{As+B}{(as^2+bs+c)^r}$ $(r = 1, 2, \ldots)$ and by finding the Laplace transform of each of the partial fractions, we can find $L^{-1}\{\bar{f}(s)\}$

Example :—*Find the inverse Laplace transform of each of the following functions.*

(i) $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (ii) $\dfrac{3s+1}{(s-1)(s^2+1)}$

(iii) $\dfrac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3}$ (iv) $\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$

6s^2+11s-6

(i) $$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$$

$$= \frac{1/2}{s-1} - \frac{1}{s-2} + \frac{5/2}{s-3}$$

(s-1)(s-2)

From the table, we have

$$L^{-1}\left\{\frac{1}{s-1}\right\} = e^{t},\ L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t},\ L^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$$ s^2

Therefore

$$f(t) = L^{-1}\left\{\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right\}$$

$s^3 - 6s^2 + 11s - 6$

$= s^3 + 11s - 6s^2 - 6$

$s(s+11) - 6(s^2 \ldots$

$$= \frac{1}{2} L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2} L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \frac{1}{2} e^{t} - e^{2t} + \frac{5}{2} e^{3t}$$

(ii) $$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} - \frac{2s-1}{s^2+1}$$

$$f(t) = L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{2}{s-1}\right\} - L^{-1}\left\{\frac{2s-1}{s^2+1}\right\}$$

$$= L^{-1}\left\{\frac{2}{s-1}\right\} - 2L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2e^{t} - 2\cos t + \sin t$$

(iii) $$\frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3} = \frac{2}{s+1} + \frac{4}{s-2} + \frac{3}{(s-2)^2} - \frac{1}{(s-2)^3}$$

$e^{t}-2c$

From the table and using shifting theorem

$$L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{t^{n-1}}{(n-1)!} e^{at}$$

$\frac{1}{(S+3)^2}$

Hence

$$f(t) = L^{-1}\left\{\frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3}\right\}$$

$$= 2L^{-1}\left\{\frac{1}{s+1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} + 3L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$- L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$e^{-2t} \times \frac{t^4}{2}$

$= \frac{e^{-2t} t^3}{2}$

$$= 2e^{-t} + 4e^{2t} + 3te^{2t} - \frac{t^2}{2!} e^{2t}$$

$$= 2e^{-t} + (4 + 3t - \tfrac{1}{2}t^2)\, e^{2t}$$

(iv) The quadratic factors cannot be resolved into real factors with real numbers, hence

$$\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \frac{3}{s^2+2s+5} - \frac{2}{s^2+2s+2}$$

$$= \frac{(3/2).2}{(s+1)^2+(2)^2} - \frac{2}{(s+1)^2+(1)^2}$$

Using shifting theorem, we have

$$f(t) = L^{-1}\left\{\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\}$$

$$= \frac{3}{2} \cdot L^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\} - 2L^{-1}\left\{\frac{1}{(s+1)^2+(1)^2}\right\}$$

$$= \frac{3}{2} e^{-t} \sin 2t - 2e^{-t} \sin t$$

(*III*) If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0) = 0$, then

$$L^{-1}\{s\,\bar{f}(s)\} = f'(t)$$

i. e. if known standard transform $\bar{f}(s)$ is multiplied by s, the inverse transform is the differentiation of $f(t)$.

This can be generalised as

$$\boxed{L^{-1}\left\{s^n \bar{f}(s)\right\} = \frac{d^n}{dt^n}\left\{f(t)\right\}} \qquad \ldots \quad \ldots \quad (24)$$

with conditions $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$

Sometimes along with result (24), we require to use the result (17) i. e.

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

$$= (-1)^n \bar{f}^{(n)}(s)$$

which can be expressed as

$$\boxed{L^{-1}\left\{\bar{f}^{(n)}(s)\right\} = (-1)^n t^n f(t)} \qquad \ldots \quad \ldots \quad (25)$$

Example 1 :— Find the inverse Laplace transform of the following

(i) $\frac{s^2}{(s^2+a^2)^2}$ (ii) $\frac{s^2}{(s+a)^3}$ (iii) $\log\left(1+\frac{a^2}{s^2}\right)$

(iv) $\tan^{-1}\left(\frac{2}{s^2}\right)$

(i) $\bar{f}(s) = \frac{1}{s^2+a^2}$

$$\therefore L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at = f(t)$$

Now $\bar{f}'(s) = \frac{-2s}{(s^2+a^2)^2}$

Hence using result (25), we get

$$L^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\} = (-1)^{(1)}\, t\, \frac{1}{a} \sin at$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\, t \sin at$$

Now using result (24), we have

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{s \cdot \frac{s}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{2a}\, t \sin at\right\}$$

$$= \frac{1}{2a}\,(at \cos at + \sin at)$$

(ii) We have, if

$$\bar{f}(s) = \frac{1}{s+a}, f(t) = e^{-at}$$

$$\text{Now } \bar{f}'(s) = -\frac{1}{(s+a)^2}, \bar{f}''(s) = \frac{2}{(s+a)^3}$$

ence by result (25)

$$L^{-1}\left\{\frac{2}{(s+a)^3}\right\} = (-1)^2\, t^2\, e^{-at}$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s+a)^3}\right\} = \frac{1}{2}\, t^2\, e^{-at}$$

Using result (24)

$$L^{-1}\left\{\frac{s^2}{(s+a)^3}\right\} = \frac{d^2}{dt^2}\left\{\frac{1}{2}\, t^2\, e^{-at}\right\}$$

$$= \frac{1}{2}\left[a^2 t^2 - 4at + 2\right] e^{-at}$$

(iii) $$\bar{f}(s) = \log\left(1+\frac{a^2}{s^2}\right)$$

$$= \log(s^2+a^2) - 2\log s$$

$$\bar{f}'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s} = F(s)$$

By result (25),

$$\therefore \quad L^{-1} f'(s) = 2(\cos at - 1) = -t f(t)$$

$$\therefore \quad f(t) = \frac{2}{t}(1-\cos at)$$

(iv) $\bar{f}(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

$$\bar{f}'(s) = \frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right) = -\frac{4s}{s^4+4}$$

$$= -\frac{4s}{(s^2-2s+2)(s^2+2s+2)}$$

$$= -\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right]$$

$$= -\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore\ L^{-1}\bar{f}'(s) = -\left[e^{t}\sin t - e^{-t}\sin t\right]$$

$$= -\left(\frac{e^t-e^{-t}}{2}\right)2\sin t = -2\sin t\sinh t$$

$$\therefore\ L^{-1}\bar{f}'(s) = -t f(t) = -2\sin t\sinh t$$

$$\therefore\ L^{-1}\tan^{-1}\left(\frac{2}{s^2}\right) = f(t) = \frac{2}{t}\sin t\sinh t$$

(IV) Use of Convolution Theorem :

If the function $\bar{f}(s)$, whose inverse transform is required, can be expressed as a product $\bar{f}(s)\,.\bar{G}(s)$, where inverse transforms of $\bar{F}(s)$ and $\bar{G}(s)$ are known, we can use convolution theorem viz.

If $L^{-1}\bar{F}(s) = F(t)$, $L^{-1}\bar{G}(s) = G(t)$ and

$$\bar{f}(s) = \bar{F}(s)\,.\,\bar{G}(s)$$

then

$$\boxed{\left\{L^{-1}\bar{f}(s)\right\} = L^{-1}\left\{\bar{F}(s)\,.\,\bar{G}(s)\right\} = \int_0^t F(t-u)\,G(u)\,du} \quad \dots (26)$$

Corollary :—

Since $L^{-1}\left(\frac{1}{s}\right) = 1$ and $L^{-1}\bar{f}(s) = f(t)$

Let $\bar{F}(s) = \dfrac{1}{s}$ and $\bar{G}(s) = \bar{f}(s)$, hence by result (26), we get

$$\boxed{L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t 1 \cdot f(u)\, du} \tag{27}$$

Note :— F (t) and G (t) are interchangeable.

Example 1 :—Obtain the inverse Laplace transform of the following :

(i) $\dfrac{1}{s^2(s+1)^2}$ (ii) $\dfrac{s}{(s^2+a^2)^2}$

(i) $\dfrac{1}{s^2(s+1)^2} = \dfrac{1}{s^2} \quad \dfrac{1}{(s+1)^2}$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = t \quad \text{and} \quad L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t e^{-t}$$

$$= F(t) \qquad\qquad = G(t)$$

Hence using result (26), we have

$$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t (t-u)\, u e^{-u} du$$

$$= \left[-(ut-u^2)e^{-u} - (t-2u)e^{-u} - (-2)e^{-u}\right]_0^t$$

$$= t e^{-t} + 2e^{-t} + t - 2.$$

Aliter :— We can use result (27) repeatedly, thus

$$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t e^{-t} = f(t) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

By result (27)

$$L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{(s+1)^2}\right\} = \int_0^t u e^{-u} du \quad [\text{from (i)}]$$

$$= \left[-u e^{-u} - e^{-u}\right]_0^t$$

$$= 1 - (t+1)e^{-t} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Using result (27), we have

$$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s+1)^2}\right\}$$

$$= \int_0^t \left[1-(u+1)\,e^{-u}\right] du$$

$$= \left[u + (u+1)\,e^{-u} + e^{-u}\right]_0^t$$

$$= te^{-t} + 2e^{-t} + t - 2$$

(ii) $$\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = F(t)$$ (Cos a(t-u))

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} = G(t)$$ $\frac{\sin au \; du}{a}$

Hence by result (26)

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}\right\}$$

$$= \int_0^t \cos a(t-u)\,\frac{\sin au}{a}\,du$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin(2au - at)\right] du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos(2au - at)\right]_0^t$$

$$= \frac{1}{2a}\,t\sin at$$ [This problem can be solved by using result (25)]

Example 2 :—Find the inverse transform of the following.

(i) $\dfrac{1}{(s-2)^4(s+3)}$ ii) $\dfrac{1}{s\sqrt{s+4}}$

(iii) $\dfrac{1}{s}\log\dfrac{s+3}{s+2}$

(i) $L^{-1}\dfrac{1}{(s-2)^4(s+3)} = L^{-1}\dfrac{1}{(s-2)^4(s-2+5)}$

Note the step

$$= e^{2t} L^{-1}\frac{1}{s^4(s+5)}$$

By convolution theorem :

$$L^{-1}\frac{1}{s^4} = \frac{t^3}{3!} \quad , \quad L^{-1}\frac{1}{s+5} = e^{-5t}$$

$$\therefore L^{-1}\frac{1}{s^4(s+5)} = \int_0^t \frac{u^3}{6} e^{-5(t-u)} du$$

$$= \frac{e^{-5t}}{6}\int_0^t u^3 e^{5u}\, du$$

$$= \frac{e^{-5t}}{6}\left[\left(\frac{1}{5}u^3 - \frac{3}{25}u^2 + \frac{6}{125}u - \frac{6}{625}\right)e^{5u}\right]_0^t$$

$$= \frac{1}{6}\left[\frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625}\right] + \frac{e^{-5t}}{625}$$

(ii) $L^{-1}\left(\dfrac{1}{s}\right) = 1,\ L^{-1}\dfrac{1}{\sqrt{s+4}} = e^{-4t} L^{-1}\left(\dfrac{1}{\sqrt{s}}\right) = e^{-4t}\dfrac{t^{-1/2}}{\Gamma(\frac{1}{2})}$

$$= \frac{e^{-4t}}{\sqrt{\pi t}}$$

Hence by convolution theorem

$$L^{-1}\frac{1}{s\sqrt{s+4}} = \int_0^t \frac{e^{-4x}}{\sqrt{\pi x}} dx = \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-y^2} dy, \quad (4x = y^2)$$

$$= \frac{1}{2} erf\,(2\sqrt{t})$$

$$\text{since } erf\,(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2} dx$$

(iii) Let $\overline{f}(s) = \log\dfrac{s+3}{s+2}$

$$\therefore \overline{f}'(s) = \frac{1}{s+3} - \frac{1}{s+2}$$

$$\therefore L^{-1}\bar{f}'(s)(t) = e^{-3t} - e^{-2t} = -t f(t)$$

$$\therefore f(t) = \frac{e^{-2t} - e^{-3t}}{t}$$

$$L^{-1}\left[\frac{1}{s}\bar{f}(s)\right] = \int_0^t \frac{e^{-2s} - e^{-3s}}{s}\, ds.$$

Example 3. Use convolution theorem to prove that

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Consider the function

$$f(x) = \int_0^t f_1(x) f_2(t-x)\, dx \qquad \ldots \quad \ldots \quad \ldots \quad (i)$$

where $t = 1$ and

$$f_1(x) = x^{m-1} \;\therefore\; \bar{f}_1(s) = \frac{\Gamma(m)}{s^m} \qquad \ldots \quad \ldots \quad \ldots \quad (ii)$$

$$f_2(x) = x^{n-1} \;\therefore\; \bar{f}_2(s) = \frac{\Gamma(n)}{s^n} \qquad \ldots \quad \ldots \quad \ldots \quad (iii)$$

$\therefore$ By convolution theorem

$$L f(x) = \bar{f}_1(s) \cdot \bar{f}_2(s)$$

$$= \frac{\Gamma(m)\Gamma(\;)}{s^{m+n}}$$

$$\therefore f(x) = \int_0^t f_1(x) f_2(t-x)\, dx$$

$$= L^{-1}\frac{\Gamma(m)\Gamma(n)}{s^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\, t^{m+n-1}$$

Putting $t = 1$, we get

$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

For Laplace transforms to be used effectively in solving differential equations, it is necessary to have a good practice with the methods of getting inverse Laplace transforms. Many standard books on Laplace transform provide an extended Table of Laplace transforms and by reference to it, the Inverse Laplace Transform is easily obtained.

Invese Laplace Transform Table

$\bar{f}(s)$	$L^{-1}\bar{f}(s) = f(t)$
$\frac{1}{s-a}$ $\frac{1}{s}$	e^{at} 1
$\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma(n+1)}$ $\frac{t^n}{n!}$ (n + ve integre).
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\bar{f}(s-a)$	$e^{at} f(t)$
$e^{-as}\bar{f}(s)$	$F(t) \le f(t-a),\ t > a$ $= 0, \quad t < a$
$\bar{f}(s/a)$	$a f(at)$
$s\bar{f}(s)$	$f'(t)$, if $f(0) = 0$
$(-1)^n \frac{d^n}{ds^n}\bar{f}(s)$	$t^n f(t)$
$\frac{1}{s}\bar{f}(s)$	$\int_0^t f(t)\,dt$
$\int_s^\infty \bar{f}(u)\,du$	$\frac{1}{t} f(t)$
$\bar{f}_1(s)\bar{f}_2(s)$	$\int_0^t f_1(u) f_2(t-u)\,du$

Examples XIII-B

Find the inverse Laplace transform of each of the following functions :-

Ex. 1. (i) $\frac{1}{2s-3}$ (ii) $\frac{1}{(s-1)^5}$ (iii) $\frac{3s+5\sqrt{2}}{s^2+8}$

(iv) $\frac{4s+15}{16s^2-25}$ (v) $\frac{3(s^2-1)^2}{2s^5}$

(vi) $\frac{1}{s^{3/2}}$ (vii) $\frac{1}{\sqrt{2s+3}}$ (viii) $\frac{s+1}{s^{4/3}}$

(ix) $\left(\frac{1-\sqrt{s}}{s^2}\right)^2$ (x) $\frac{1}{(s+4)^{5/2}}$ (xi) $\frac{8e^{-3s}}{s^2+4}$

(xii) $\frac{e^{-2s}}{s^2+8s+25}$ (xiii) $\frac{e^{-as}}{(s+b)^{5/2}}$ (xiv) $\left(\frac{1-\sqrt{s}}{s^2}\right)^2 e^{-s}$

[Ans. (i) $\frac{1}{2}e^{3/2\,t}$ (ii) $\frac{1}{24}t^4e^{-t}$ (iii) $3\cos 2\sqrt{2}\,t+\frac{5}{2}\sin 2\sqrt{2}\,t$

(iv) $\frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$ (v) $\frac{3}{2}-\frac{3}{2}t^2+\frac{1}{16}t^4$

(vi) $2\sqrt{\frac{t}{\pi}}$ (vii) $\frac{1}{\sqrt{2\pi t}}e^{-3t/2}$ (viii) $\frac{1}{\Gamma(\frac{1}{3})}\left(t^{-2/3}+3t^{1/3}\right)$

(ix) $\frac{t^3}{6}+\frac{t^2}{2}-\frac{16}{15\sqrt{\pi}}t^{5/2}$ (x) $\frac{2}{\sqrt{\pi}}e^{-4t}\sqrt{t}$

(xi) $f(t)=4\sin 2(t-3),\ t>3$ (xii) $f(t)=\frac{1}{3}e^{-4(t-2)}\sin 3(t-2),\ t>2$

$=0 \qquad t<0 \qquad\qquad =0,\ t<2$

(xiii) $f(t)=\frac{4}{3\sqrt{\pi}}e^{-b(t-a)}(t-a)^{3/2},\ t>a$

$=0,\quad t<a$

(xiv) $f(t)=\left[\frac{(t-1)^3}{6}+\frac{(t-1)^2}{2}-\frac{16}{15\sqrt{\pi}}(t-1)^{5/2}\right],\ t>1$

$=0,\quad t<1$

Ex. 2. (1) $\frac{4s+12}{s^2+8s+16}$ [Ans. $4e^{-4t}(1-t)$]

(2) $\frac{s+1}{s^2+s+1}$ [Ans. $\frac{1}{\sqrt{3}}e^{-t/2}\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}t+\sin\frac{\sqrt{3}}{2}t\right)$]

(3) $\frac{3s+7}{s^2-2s-3}$ [Ans. $4e^{3t}-e^{-t}$]

(4) $\dfrac{1}{s(s+1)(s+2)(s+3)}$ $\left[\text{Ans. } \dfrac{1}{6} - \dfrac{1}{2}e^{-t} + \dfrac{1}{2}e^{-2t} - \dfrac{1}{6}e^{-3t}\right]$

(5) $\dfrac{s-1}{s^2+2s}$ $\left[\text{Ans. } -\dfrac{1}{2} + \dfrac{3}{2}e^{-2t}\right]$

(6) $\dfrac{s^2+1}{s^3+3s^2+2s}$ $\left[\text{Ans. } \dfrac{1}{2} - 2e^{-t} + \dfrac{5}{2}e^{-2t}\right]$

(7) $\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}$ $\left[\text{Ans. } \dfrac{3}{50}e^{3t} - \dfrac{1}{25}e^{-2t} - \dfrac{1}{50}e^{-t}(\cos 2t - 18\sin 2t)\right]$

(8) $\dfrac{s+29}{(s+4)(s^2+9)}$ $\left[\text{Ans. } e^{-4t} - \cos 3t + \dfrac{5}{3}\sin 3t\right]$

(9) $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$ $\left[\text{Ans. } -\dfrac{1}{6}e^{-t} - \dfrac{4}{3}e^{2t} + \dfrac{7}{2}e^{3t}\right]$

(10) $\dfrac{s+2}{s^2(s-1)^2}$ $\left[\text{Ans. } (3t-8)e^{t} + t^2 + 5t + 8\right]$

(11) $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ $\left[\text{Ans. } \dfrac{1}{3}e^{-t}(\sin t + \sin 2t)\right]$

(12) $\dfrac{1}{s^3(s^2+1)}$ $\left[\text{Ans. } \dfrac{1}{2}t^2 + \cos t - 1\right]$

(13) $\dfrac{21s-33}{(s+1)(s-2)^3}$ $\left[\text{Ans. } 2e^{-t} - 2e^{2t} + 6te^{2t} + \dfrac{3}{2}t^2e^{2t}\right]$

(14) $\dfrac{s^2}{(s^2+1)^2}$ $\left[\text{Ans. } \dfrac{1}{2}(t\cos t + \sin t)\right]$

(15) $\dfrac{s+2}{s^2(s+3)}$ $\left[\text{Ans. } \dfrac{2}{3}t + \dfrac{1}{9} - \dfrac{1}{9}e^{-3t}\right]$

(16) $\dfrac{s+1}{(s^2+2s+2)^2}$ $\left[\text{Ans. } \dfrac{1}{2}te^{-t}\sin t\right]$

(17) $\dfrac{1}{s^3+a^3}$ $\left[\text{Ans. } \dfrac{1}{3a^2}\left\{e^{-at} - e^{\frac{at}{2}}\left(\cos\dfrac{\sqrt{3}}{2}at - \sqrt{3}\sin\dfrac{\sqrt{3}}{2}at\right)\right\}\right]$

(18) $\dfrac{s^2}{(s^2-a^2)^2}$ $\left[\text{Ans. } \dfrac{1}{2a}(\sinh at + at\cosh at)\right]$

(19) $\dfrac{1}{(s^2+1)^3}$ $\left[\text{Ans. } \dfrac{1}{8}\{(3-t^2)\sin t - 3t\cos t\}\right]$

(20) $\dfrac{s^2-a^2}{(s^2+a^2)^2}$ $\left[\text{Ans. } t\cos at\right]$

(21) $\dfrac{3s+1}{(s+1)^4}$ $\left[\text{Ans. } e^{-t}\left(\dfrac{3}{2}t^2 - \dfrac{1}{3}t^3\right)\right]$

(22) $\dfrac{s}{(s^2+1)(s^2+4)}$ $\left[\text{Ans. } \dfrac{1}{3}(\cos t - \cos 2t)\right]$

(23) $\dfrac{s+2}{(s+3)(s+1)^3}$ $\left[\text{Ans. } \dfrac{1}{8}(2t^2+2t-1)e^{-t} + \dfrac{1}{8}e^{-3t}\right]$

(24) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ $\left[\text{Ans. } \dfrac{1}{4}e^{-2t}(2t\cos 2t + \sin 2t)\right]$

(25) $\dfrac{1}{(s^2+a^2)^2}$ $\left[\text{Ans. } \dfrac{1}{2a^3}(\sin at - at\cos at)\right]$

(26) $\dfrac{s}{s^4+4a^4}$ $\left[\text{Ans. } \dfrac{1}{2a^2}\sin at \sinh at\right]$

(27) $\dfrac{s}{s^4+s^2+1}$ $\left[\text{Ans. } \dfrac{2}{\sqrt{3}}\sin\dfrac{\sqrt{3}}{2}t \sinh\dfrac{t}{2}\right]$

3. Use convolution theorem to obtain inverse Laplace transform of each of the following functions :—

(i) $\dfrac{a}{s(s-a)}$ $\left[\text{Ans. } e^{at}-1\right]$

(ii) $\dfrac{1}{s(s^2+a^2)}$ $\left[\text{Ans. } \dfrac{1}{a^2}(1-\cos at)\right]$

(iii) $\dfrac{1}{(s-2)(s+2)^2}$ $\left[\text{Ans. } \dfrac{1}{16}\left(e^{2t}-e^{-2t}-4t\,e^{-2t}\right)\right]$

(iv) $\dfrac{1}{(s+1)(s^2+1)}$ $\left[\text{Ans. } \dfrac{1}{2}\left(\sin t - \cos t + e^{-t}\right)\right]$

(v) $\dfrac{s^2}{(s^2+4)^2}$ $\left[\text{Ans. } \dfrac{1}{2}t\cos 2t + \dfrac{1}{4}\sin 2t\right]$

4. If s is sufficiently large, show, using series expansion of $\tan^{-1}\left(\dfrac{a}{s}\right)$, that

$$L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\} = \frac{\sin at}{t}$$

5. Find $f(t)$, if $\bar{f}(s)$ is given by

(i) $\log\left(\dfrac{s+b}{s+a}\right)$ (ii) $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$

(iii) $\dfrac{1}{2}\log\left(\dfrac{s-1}{s+1}\right)$ (iv) $\dfrac{1}{2}\log\left(\dfrac{s^2-a^2}{s^2}\right)$

[Ans. : (i) $\dfrac{1}{t}(e^{-at}-e^{-bt})$ (ii) $\dfrac{1}{t}(\cos at - \cos bt)$

(iii) $\dfrac{1}{t}\sinh t$ (iv) $\dfrac{1}{t}(1-\cosh at)$]

6. Show that

(i) $L^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots$

(ii) $L^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\} = \frac{1}{2a}$ (cosh at sin at + sinh at cos at)

(iii) $L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\} = \int_0^t \frac{2}{x}(1-\cos x)\,dx$

(iv) If $L\{J_0(x)\} = \frac{1}{\sqrt{s^2+1}}$, then

$$\int_0^t J_0(x)\,J_0(t-x)\,dx = \sin t$$

13.6. Solution of Ordinary Linear Differential Equations with Constant Coefficients :

We are now in a position to develop a method for solving linear differential equations with constant coefficients, when the initial (or the boundary) conditions are given.

Let us first take a simple problem, which will bring out the peculiarities of the Laplace transform method

Ex.1. Solve the differential equation

$$(D^2+1)\,y = 0,\ t > 0,\ D \equiv \frac{d}{dt}$$

with $y = 1$, $Dy = 2$ when $t = 0$.

If we multiply the equation through by e^{-st} and integrate from 0 to ∞, we get the Laplace transform of the whole equation. That is we have

$$L\left\{\frac{d^2y}{dt^2} + y\right\} = 0$$

or $L\left\{\frac{d^2y}{dt^2}\right\} + L\{y\} = 0$ .. .. .. (i)

Now $L\left\{\frac{d^2y}{dt^2}\right\} = s^2\bar{y}(s) - sy_0 - y_1$ [by result 23)]

$$L\{y\} = \bar{y}(s)$$

and as $y_0 = (y)_{t=0} = 1$, $y_1 = (Dy)_{t=0} = 2$,

$$\left.\begin{aligned} \text{Therefore } L\left(\frac{d^2y}{dt^2}\right) &= s^2\bar{y}(s) - s - 2 \\ \text{and } L(y) &= \bar{y}(s) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Using (ii) in (i), equation (i) becomes,

$$\bar{y}(s)\,[s^2 + 1] = s + 2 \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

This is known as a *subsidiary equation* and it is an algebraic equation; from which $\bar{y}(s)$ is obtained as

$$\bar{y}(s) = \frac{s+2}{s^2+1} = \frac{s}{s^2+1} + 2\,\frac{1}{s^2+1} \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

Knowing $\bar{y}(s)$, using the Table of transforms, we obtain $y(t)$ as

$$y(t) = \cos t + 2 \sin t$$

which is the solution of the given differential equation.

Thus the Laplace transform method changes a differential equation with initial values to an algebraic equation (the subsidiary equation) from which $\bar{y}(s)$ is known and its inverse Laplace transform $y(t)$ is the solution of the given differential equation. Another peculiarity of this method is that the initial conditions are included in the method from the very beginning, and they are not imposed at the end of the solution as is done in other methods of solving differential equations.

Let us now develop a routine process for transforming a given linear differential equation into a subsidiary equation.

Consider the differential equation

$$D^n y + a_1 D^{n-1} y + \ldots + a_{n-1} Dy + a_n y = f(t),\ t > 0 \quad \ldots \quad (28)$$

with initial values of $y, Dy, \ldots D^{n-1}y$ as

$y_0, y_1, \ldots y_{n-1}$ respectively, when $t = 0$.

Taking the Laplace transform of the equation (28), we get

$$L\,[D^n y + a_1 D^{n-1} y + \ldots + a_{n-1} Dy + a_n y] = L\,[f(t)]$$

or $$L(D^n y) + a_1 L(D^{n-1} y) + \ldots + a_{n-1} L(Dy) + a_n L(y) = L\,[f(t)] \quad \ldots \quad \ldots \quad (29)$$

With usual notation and result for Laplace transform of the derivatives, we have as

$$s^n \bar{y} - (s^{n-1} y_0 + s^{n-2} y_1 + \dots + y_{n-1})$$
$$+ a_1 s^{n-1} \bar{y} - a_1 (s^{n-2} y_0 + s^{n-3} y_1 + \dots + y_{n-2})$$
$$+ \dots + a_{n-1} s\bar{y} - a_{n-1} y_0 + a_n \bar{y} = \bar{f}(s) \quad \dots (30)$$

where $\bar{y} = \bar{y}(s) = L[y]$ and $\bar{f}(s) = L[f(t)]$.

We shall group the terms in (30) which is the subsidiary equation and rewrite (30) as

$$(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n)\bar{y}$$
$$= \bar{f}(s) + a_{n-1} y_0 + a_{n-2}(s y_0 + y_1) + \dots$$
$$\dots + (s^{n-1} y_0 + s^{n-2} y_1 + \dots + y_{n-1}) \quad \dots (31)$$

This now helps us to form a rule to write the subsidiary equation immediately.

(a) The L. H. S. of (31) is obtained by replacing $D^r y$ by $s^r \bar{y}$, $(r = 0, 1, \dots n)$ in the L. H. S. of the differential equation (28).

(b) The R. H. S. is obtained by writing $\bar{f}(s)$, [the L. T. of $f(t)$, the function on r. h. s. of differential equation (28)] and adding the terms to it as follows :

For Dy on L. H. S. of (28) add y_0 on R. H. S. of subsidiary equation.

,,	$D^2 y$	,,	add $s y_0 + y_1$,,
,,	$D^3 y$	,,	add $s^2 y_0 + s y_1 + y_2$
......			......
For	$D^r y$	,,	add $s^{r-1} y_0 + s^{r-2} y_1 + \dots + y_{r-1}$

and so the subsidiary equation (25) is formed. From the subsidiary equation $\bar{y}(s)$ is found and hence $y(t)$, is the solution of the differential equation.

Let us see this routine technique applied to one or two problems.

Example 1 :— Solve the equation

$$(D^2+5D+6)\,y=1,\ t>0$$

with initial conditions $y=y_0$, $Dy=y_1$ at $t=0$.

The L. H. S. of the subsidiary equation is obtained by writting s for D in L. H. S. of the equation is

$$(s^2+5s+6)\,\bar{y}\,(s)$$

To form the R. H. S. of the subsidiary equation, we see that

$$f\,(t)=1,\ \text{and}\ \bar{f}\,(s)=\frac{1}{s}.$$

The R. H. S. of subsidiary equation

$$=\bar{f}\,(s)+\text{terms added as follows.}$$

Corresponding to D^2y present in L. H. S. of dift. equation we have to add sy_0+y_1 and corresponding to Dy in L. H. S. of dift. equation, we have y_0.

$\therefore$ The R. H. S. of subsidiary equation is

$$\frac{1}{s}+(sy_0+y_1)+5\,(y_0).$$

The subsidiary equation of the given differential equation is

$$(s^2+5s+6)\,\bar{y}\,(s)=\frac{1}{s}+sy_0+y_1+5y_0$$

From which

$$\bar{y}\,(s)=\frac{1}{s(s+2)\,(s+3)}+\frac{sy_0+y_1+5y_0}{(s+2)\,(s+3)}$$

$$=\frac{1}{6}\cdot\frac{1}{s}+\frac{1}{3}\cdot\frac{1}{s+3}-\frac{1}{2}\cdot\frac{1}{s+2}+\frac{sy_0+y_1+5y_0}{(s+2)\,(s+3)}$$

$$=\frac{1}{6}\cdot\frac{1}{s}+\frac{1}{3}\cdot\frac{1}{s+3}-\frac{1}{2}\cdot\frac{1}{s+2}+\frac{y_1+3y_0}{s+2}-\frac{y_1+2y_0}{s+3}$$

and so using the Table of transforms, we get the solution

$$y\,(t)=\frac{1}{6}+\frac{1}{3}e^{-3t}-\frac{1}{2}e^{-2t}+(y_1+3y_0)\,e^{-2t}-(y_1+2y_0)\,e^{-3t}.$$

$$=\frac{1}{6}+\left(3y_0+y_1-\frac{1}{2}\right)e^{-2t}+\left(\frac{1}{3}-y_1-2y_0\right)e^{-3t}.$$

Example 2. **Solve**

$$\frac{d^2y}{dt^2}+9y=18t,\ \text{ if } y\,(0)=0 \text{ and } y\left(\frac{\pi}{2}\right)=0.$$

Note in this example $y'\,(0)$ is not given, hence we assume $y'\,(0)=A$.

Taking the Laplace transform of the equation, we have

$$L\left\{\frac{d^2y}{dt^2} + 9y\right\} = L(18t)$$

$$(s^2 + 9)\,\bar{y}(s) = \frac{18}{s^2} + s\,y(0) + [\,y'(0) + A\,]$$

[Using result (31)]

$$\therefore \bar{y}(s) = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$$

$$= 18\left\{\frac{1}{9}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right)\right\} + \frac{A}{s^2+9}$$

$$= \frac{2}{s^2} + \frac{A-2}{s^2+9}$$

$$\therefore y(t) = 2t + \frac{1}{3}(A-2)\sin 3t$$

To determine A, we use $y\left(\frac{\pi}{2}\right) = 0$

$$\therefore 0 = \pi - \frac{1}{3}(A-2)$$

$$\therefore A = 3\pi + 2$$

$\therefore$ The solution is

$$y(t) = 2t + \frac{1}{3}[(3\pi+2) - 2]\sin 3t$$

$$= 2t + \pi \sin 3t.$$

13.7. Solution of Simultaneous Ordinary Differential Equations :

The Laplace transform method is very useful to solve two or more simultaneous ordinary differential equations. The procedure is the same as that used in art. 7.5. Here the method transforms the equations into solution of simultaneous algebraic equations.

Example : Solve the equations

$$\left.\begin{aligned} \frac{dx}{dt} + \frac{dy}{dt} + x &= -e^{-t} \\ \frac{dx}{dt} + 2\frac{dy}{dt} + 2x + 2y &= 0 \end{aligned}\right\} \quad \ldots \quad \text{(i)}$$

subject to the condition $x(0) = -1$, $y(0) = 1$.

Taking the Laplace transform of the equation in (i), we get

[using notation $\bar{x}(s) = \bar{x}$, $\bar{y}(s) = \bar{y}$]

$$\left.\begin{aligned} [s\bar{x}+1] + [s\bar{y}-1] + \bar{x} &= -\frac{1}{s+1} \\ [s\bar{x}+1] + 2[s\bar{y}-1] + 2\bar{x} + 2\bar{y} &= 0 \end{aligned}\right\}$$

i. e.

$$\left.\begin{aligned} (s+1)\bar{x} + s\bar{y} &= -\frac{1}{s+1} \\ [s+2]\,\bar{x} + 2[s+1]\bar{y} &= 1 \end{aligned}\right\} \quad \ldots \ldots \ldots \text{(ii)}$$

Solving the simultaneous algebraic equations (ii) for unknows $\bar{x}$ and $\bar{y}$

we get

$$\bar{x}(s) = -\frac{s+2}{s^2+2s+2}$$

$$= -\left\{\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right\} \quad \ldots \ldots \ldots \text{(iii)}$$

and

$$\bar{y}(s) = \frac{s^2+3s+3}{(s^2+2s+2)(s+1)}$$

$$= \frac{1}{s+1} + \frac{1}{(s+1)^2+1} \quad \ldots \ldots \ldots \ldots \ldots \text{(iv)}$$

Taking inverse Laplace transforms of (iii) and (iv), the solutions of the equations are

$$x(t) = -e^{-t}[\cos t + \sin t]$$

$$y(t) = e^{-t}[1 + \sin t]$$

Examples XIII—C

Solve the following differential equations ($t>0$) with given initial values

$(D+1)^2 y = \sin t$, with $y = \dfrac{dy}{dt} = 1$ at $t = 0$.

$$\left[\text{Ans. } y = \frac{5}{2} t e^{-t} + \frac{3}{2} e^{-t} - \frac{1}{2}\cos t\right]$$

(2) $(D^2+4D+8)y = 1$, with $y = 0$, $Dy = 1$ at $t = 0$.

$$\left[\text{Ans. } y = \frac{1}{8}\left(1 - e^{-2t}\cos 2t - 3e^{-2t}\sin 2t\right)\right]$$

(3) $(D+1)\,y = t^2 e^{-t}$, given $y = 3$ when $t = 0$

$$\left[\text{Ans. } y(t) = e^{-t}\left(\frac{t^3}{3} + 3\right)\right]$$

(4) $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 6y = 2$, with $y(0) = 1,\; y'(0) = 0$.

$$\left[\text{Ans. } y = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}\right]$$

(5) $\dfrac{d^2y}{dt^2} + y = \sin t$ with $y(0) = 1,\; y'(0) = -\dfrac{1}{2}$.

$$\left[\text{Ans. } y = \left(1 - \frac{t}{2}\right)\cos t\right]$$

(6) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 13x = 2e^{-t}$, where $x = 0,\; \dfrac{dx}{dt} = -1$ when $t = 0$.

$$\left[\text{Ans. } x = \frac{1}{5}e^{-t} - \frac{e^{-2t}}{5}(\cos 3t + 2\sin 3t)\right]$$

(7) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - 2x = 2(1 + t - t^2)$ with $x = 0,\; \dfrac{dx}{dt} = 3$, for $t = 0$.

$$[\text{Ans. } x = t^2 - e^{-2t} + e^{t}]$$

(8) $(D^3 - 2D^2 + 5D)\,y = 0$ with $y(0) = 0, y'(0) = 0, y''(0) = 1$.

$$\left[\text{Ans. } y = \frac{1}{5} - \frac{1}{5}e^{t}\cos 2t + \frac{1}{10}e^{t}\sin 2t\right]$$

(9) $\dfrac{d^2x}{dt^2} + x = 6\cos 2t$, with $x = 3,\; \dfrac{dx}{dt} = 1$ at $t = 0$.

$$[\text{Ans. } x = 5\cos t + \sin t - 2\cos 2t]$$

(10) $(D^2 + 2D + 1)\,y = 3t\,e^{-t}, y(0) = 4, y'(0) = 2$.

$$\left[\text{Ans. } y = \left(4 + 6t + \frac{1}{2}t^3\right)e^{-t}\right]$$

(11) $(D^2 + 2D + 5)\,y = e^{-t}\sin t,\; y(0) = 0, y'(0) = 1$

$$\left[\text{Ans, } \frac{1}{3}e^{-t}(\sin t + \sin 2t)\right]$$

(12) $\dfrac{dy}{dt} + 2y + \displaystyle\int_0^t y\,dt = \sin t,\; y(0) = 1$

$$\left[\text{Ans. } y = e^{-t} - \frac{3}{2}t\,e^{-t} + \frac{1}{2}\sin t\right]$$

(13) A particle is moving along x-axis according to the law

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0.$$

If the particle is started at $x = 0$ with an initial velocity of 12 meters per sec. to the left, determine x in terms of t.

[Ans. $x = -3\, e^{-3t} \sin 4t$]

(14) A constant voltage E is applied at $t = 0$ to $L - C - R$ circuit in series with zero initial current and charge. If $n^2 = \frac{1}{LC} - \frac{R^2}{4L^2} > 0$, find the current at any time t.

$$\left[\text{Ans. } \frac{E}{nL}\, e^{-\frac{Rt}{2L}} \sin nt \right].$$

(15) Two flywheels of mt. of inertia I_1 and I_2 are connected by elastic shaft and rotating with constant angular velocity n. At $t = 0$, a constant retarding couple L is applied to the first wheel. If C is the stiffness of the shaft, find the subsequent angular velocity of the other wheel.

[Hint eqts are $I_1 \ddot{\theta}_1 - C(\theta_2 - \theta_1) = -L$, $I_2 \ddot{\theta}_2 + C(\theta_2 - \theta_1) = 0$]

$$\left[\text{Ans. } \omega_2 = n - \frac{Lt}{I_1 + I_2} + \frac{L}{K^2(I_1 + I_2)} \sin Kt,\ K^2 = C\left(\frac{1}{I_1} + \frac{1}{I_2}\right) \right]$$

(16) A particle moves along a line so that its displacemnnt x from a fixed point O at any time t is given by

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 80 \sin 5t.$$

If at $t = 0$, the particle is at rest at $x = 0$, find the displacement at any time $t > 0$.

[Ans. $x = 2\, e^{-2t} (\cos t + 7 \sin t) - 2 (\sin 5t + \cos 5t)$].

(17) Solve the simultaneous equations

(i) $\left.\begin{array}{l} \frac{dx}{dt} \text{ [illegible] } 3y \\ \frac{dy}{dt} = y - 2x \end{array}\right\}$ subject to $x(0) = 8$, $y(0) = 3$.

(ii) $$\left.\begin{aligned} \frac{dx}{dt} - \frac{dy}{dt} + 2y &= \cos 2t \\ \frac{dx}{dt} + \frac{dy}{dt} - 2x &= \sin 2t \end{aligned}\right\} \text{ subject to conditions } x(0) = 0,\ y(0) = -1.$$

(iii) $$\left.\begin{aligned} \frac{d^2x}{dt^2} + x + y &= 0 \\ 4\frac{d^2y}{dt^2} - x &= 0 \end{aligned}\right\} \begin{aligned} &x(0) = -2a,\ y(0) = -a \\ &x'(0) = 2b \\ &y'(0) = -b \end{aligned}$$

[Ans. (i) $x = 5e^{-t} + 3e^{4t},\ y = 5e^{-t} - 2e^{4t}$

(ii) $x = \frac{1}{2}e^{t}(\cos t + \sin t) - \frac{1}{2}\cos 2t$

$y = -e^{t}(\cos t - \sin t) - \sin 2t$

(iii) $x = 2a \cos \frac{t}{\sqrt{2}} + 2\sqrt{2}\, b \sin \frac{t}{\sqrt{2}}$].

(13) For the equations

$$\frac{d^2x}{dt^2} = n^2(4y - 5x),\ \frac{d^2y}{dt^2} = n^2(4x - 5y)$$

with conditions $x(0) = a,\ y(0) = 0,\ x'(0) = y'(0) = 0$, show that

$$\frac{y}{x} = \tan nt \cdot \tan 2nt.$$

LAPLACE TRANSFORMS OF SPECIAL FUNCTIONS :

In discussion of certain types of physical and engineering problems, it is required to find the solution of a differential equation of the system which it is acted on by

(a) a *periodic force or periodic voltage*

(b) a *impulsive force or voltage* acting *instantaneously* at a certain time, or a concentrated load acting at a point,

(c) a force acting on a part of the system or voltage acting for finite interval of time.

Hence in the following articles, such functions and their Laplace transforms have been discussed and problems involving such functions are solved at the end.

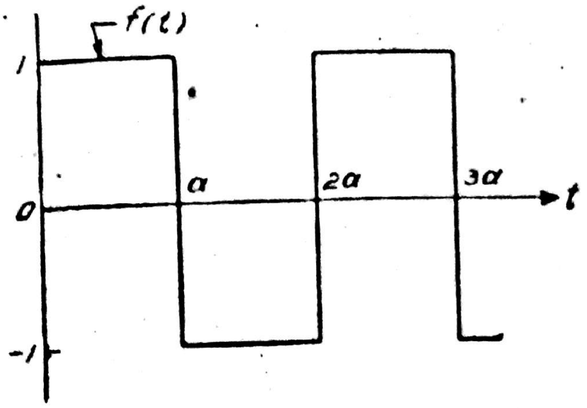
13.8 Periodic Functions :— The periodic function $f(t)$ of period T is defined as

$$f(t+T) = f(t),\ T > 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

For example, $f(t) = \sin t$ is a periodic function of period $T = 2\pi$, as

$$\begin{aligned} f(t+T) &= \sin(t + 2\pi) \\ &= \sin t = f(t) \end{aligned}$$

Another example of periodic function is a "Square Wave function for which,



$$\begin{aligned} f(t) &= 1,\ 0 \leqslant t < a \\ &= -1,\ a < t < 2a \end{aligned}$$

with a period $2a$.

Fig. 57

The Laplace transform af a periodic function $f(t)$ defined by (33) is given by

$$\bar{f}(s) = \mathrm{L}\{f(t)\}$$

$$= \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \ldots + \int^{(r+1)T} e^{-st} f(t)\,dt \quad \ldots \quad \ldots$$

$$= \sum_{r=0}^{\infty} \left[\int_{rT}^{(r+1)T} e^{-st} f(t)\,dt \right] \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

Now in the general term on R. H. S. put $t = u + rT$, i. e. $dt = du$

$$\therefore \int_{rT}^{(r+1)T} e^{-st} f(t)\, dt = e^{-rsT} \int_0^T e^{-su} f(u)\, du$$

$$[\text{ as } f(u + rT) = f(u)\ ]$$

for $r = 0, 1, 2 \ldots \ldots \ldots \ldots$

Hence from (i), we have

$$\bar{f}(s) = \sum_{r=0}^{\infty} e^{-rsT} \int_0^T e^{-su} f(u)\, du$$

$$= \left[\sum_{r=0}^{\infty} e^{-rsT}\right] \int_0^T e^{-su} f(u)\, du$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \ldots \ldots + \infty\right] \int_0^T e^{-su} f(u)\, du$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-su} f(u)\, du$$

Hence for a periodic function $f(t)$ of period T, we get

$$\boxed{\bar{f}(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-su} f(u)\, du} \quad \ldots \quad (34)$$

Example :—The "Square Wave" function of period $2a$ is defined by

$$f(t) = 1, \quad 0 < t < a$$
$$= -1, \quad a < t < 2a,$$

find the Laplace transform of $f(t)$.

By result (34), since $T = 2a$

$$L\{f(t)\} = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-su} f(u)\, du$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-su}\,1.du + \int_a^{2a} e^{-su}\,(-1)\,du\right]$$

$$= \frac{1}{s}\,\frac{(1-e^{-as})^2}{(1-e^{-2as})}$$

$$= \frac{1-e^{-as}}{1+e^{-as}}$$

$$= \frac{1}{s}\left[\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{\frac{as}{2}} + e^{-\frac{as}{2}}}\right] = \frac{1}{s}\tanh\frac{as}{2}.$$

13.9. Heaviside Unit Step Function :—One of the most useful and also simplest descontinuous function is a *unit step function* H(t), which is difined as

$$\left.\begin{aligned} H(t) &= 0, t < 0 \\ &= 1, t \geqslant 0 \end{aligned}\right\} \quad \ldots (35)$$

and is shown in the Fig. 58.

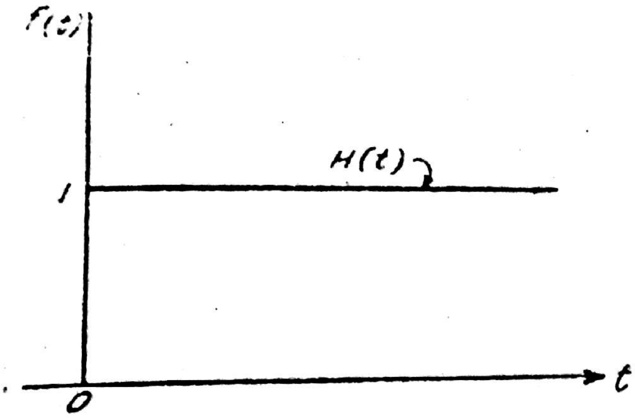


Fig. 58

Thus unit step function is a curve which has the value *zero* at all points to the left of the origin and is *unity* at all points on the right of the origin.

The *displaced unit step function* H $(t-a)$ represents the curve H (t), which is displaced to the right through a distance a to the right along the direction of t-axis, as shown in the Fig. 59. This function is defined as

$$\left.\begin{aligned} H(t-a) &= 0, t < a \\ &= 1, t \leqslant a \end{aligned}\right\} \ldots (36)$$

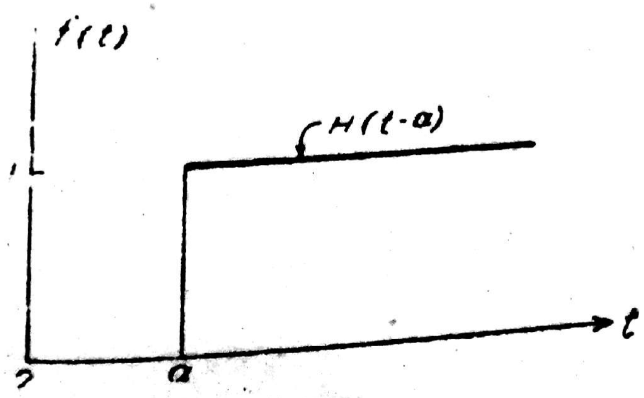


Fig. 59

Heaviside's Unit Step functions H $(t-a)$ and H (t) are extensively used to represent a portion of the curve of the function $f(t)$ as explained in the following cases.

Case 1 :—When the function $f(t)$ is multipled by *unit step function* H (t), the resultant function $f(t)$ H (t) will represent the part of the function $f(t)$ on the right of the origin, the part of f (t) on the left being cut off i. e.

$$\left.\begin{aligned} f(t)\,\mathrm{H}\,(t) &= 0,\ t<0 \\ &= f(t),\ t \geqslant 0 \end{aligned}\right\} \qquad \cdots \quad \cdots \quad \cdots \qquad (37)$$

This resultant function is illustrated in the figure 60.

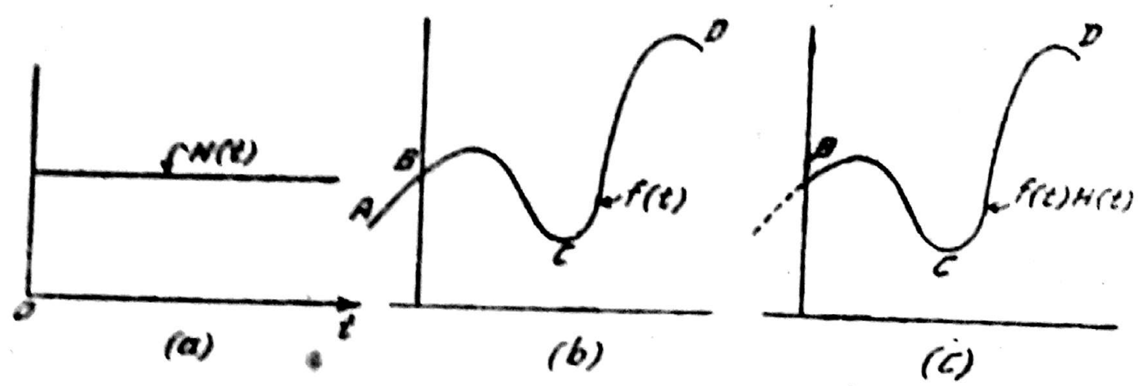


Fig. 60

Case 2 :—When the function $f(t)$ is multiplied by the displaced unit step function H $(t-a)$, the resultant function $f(t)$ H $(t-a)$ will represent the part of the function $f(t)$ on the right of $t=a$ and part of the left of $t=a$ is cut off i. e.

$$\left.\begin{aligned} f(t)\,\mathrm{H}\,(t-a) &= 0,\ t<a \\ &= f(t), t \geqslant a \end{aligned}\right\} \qquad \cdots \quad \cdots \quad \cdots \qquad (38)$$

The resultant function is shown in the Fig. 61.

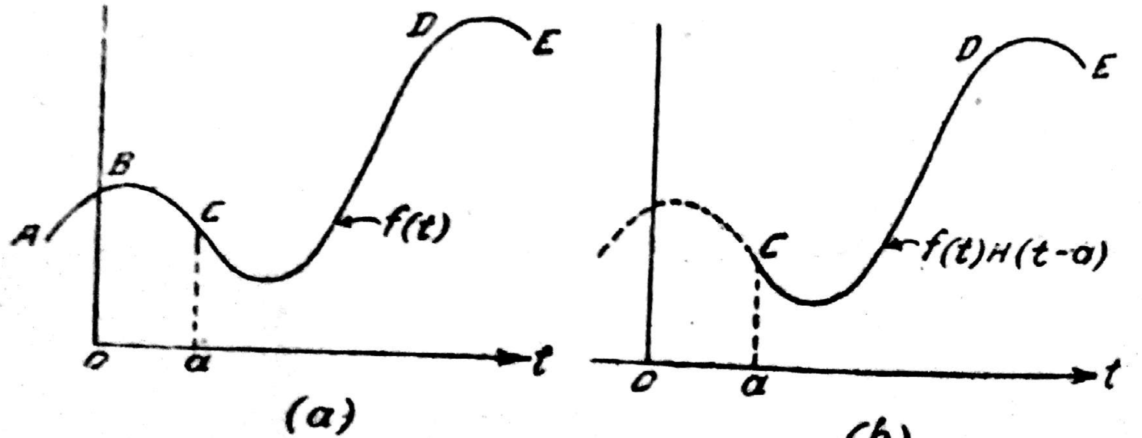


Fig. 61

Case 3 :— The *displaced function* $f(t-a)$ is obtained by shifting the curve represented by $f(t)$ [Fig. 62] to a new position along t axis, through a distance a to the right, without changing the shape and the characteristic of the function.

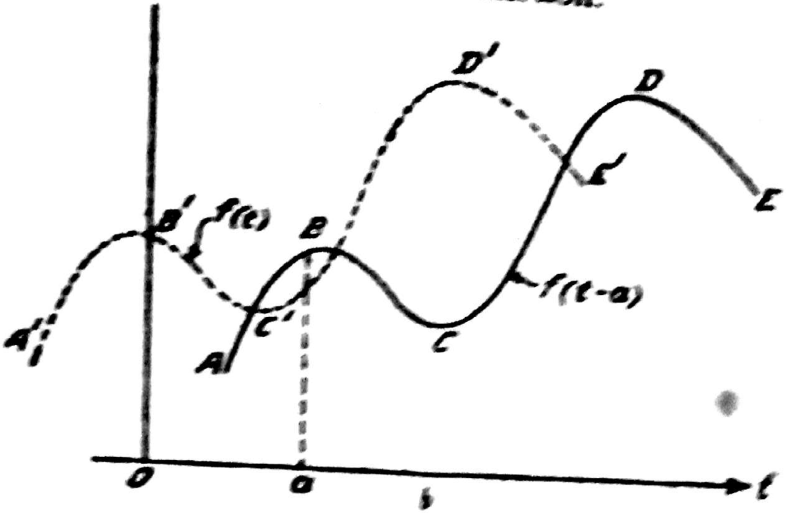


Fig. 62

When such a displaced function $f(t-a)$ is multiplied by $H(t-b)$, the resultant function $f(t-a)\,H(t-b)$ will represent the part of the function $f(t-a)$ beyond the right of $t=b$ i. e. the part CDE of the curve (thick part) in the Fig. 62 i. e.

$$\left.\begin{aligned} f(t-a)\,H(t-b) &= 0, \quad t < b \\ &= f(t-a), \quad t > b \end{aligned}\right\} \quad \ldots \quad \ldots \quad (39)$$

Case 4 :—The portion of the curve represented by $f(t)$ lying between $a \leq t \leq b$ can be expressed by using the *displaced unit step function.*

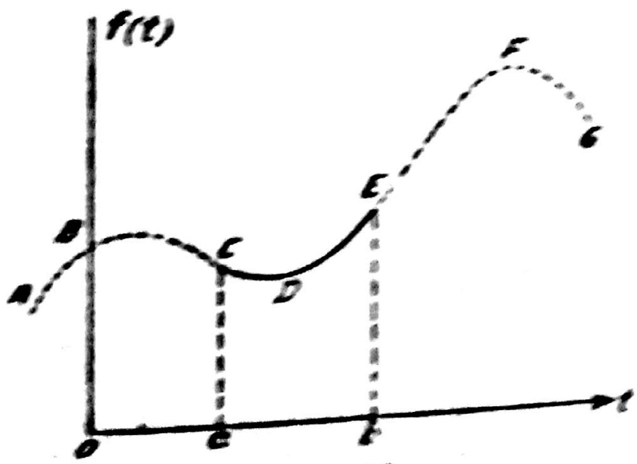


Fig. 63

The portion CDEFG ... of $f(t)$ for $t \geq a$ is given by

$$f(t)\,H(t-a) \quad \ldots \quad (i)$$

and the portion EFG ... ... of $f(t)$ beyond $t=b$ is given by

$$f(t)\,H(t-b) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Hence the part CDE (thick curve) in the Fig. 63 is given by difference of the portions of $f(t)$ represented by (i) and (ii) i. e. the part of $f(t)$ lying in the interval $a \leq t \leq b$ is given by

$$f(t)\,[H(t-a)-H(t-b)] = \left.\begin{array}{ll} 0, & t < a \\ f(t), & a \leqslant t \leqslant b \\ 0, & t > b \end{array}\right\} \quad \ldots (10)$$

The representation of a part of the function $f(t)$ in the interval $a \leqslant t \leqslant b$, is particularly useful to represent the periodic function $f(t)$ [i. e. $f(t) = f(t+T)$] in terms of a single function.

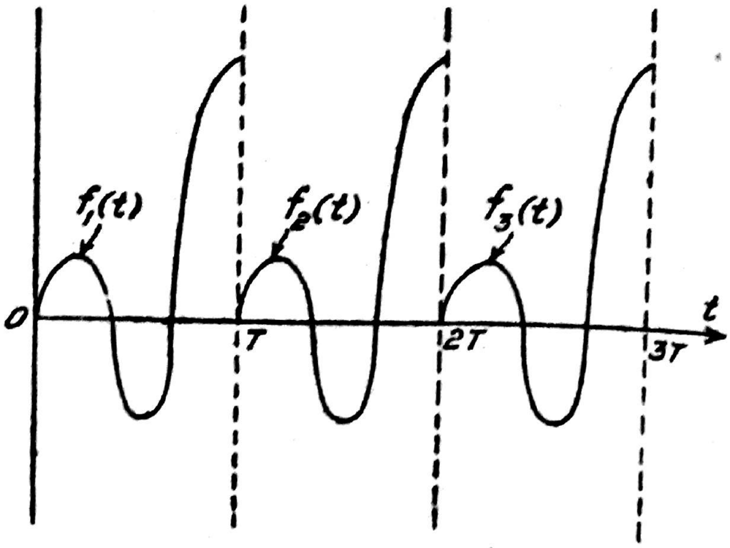


Fig. 64

The characteristic of the periodic function $f(t)$ is that, the shape of curve in $0 \leqslant t \leqslant T$ is repeated in the intervals $T \leqslant t \leqslant 2T$, $2T \leqslant t \leqslant 3T$ and so on. The curve in the interval $rT < t < (r+1)T$ is obtained by displacing the curve $f(t)$ in the interval $0 < t < T$, through a distance rT to the right along the t-axis and hence the function $f_{r+1}(t)$ in $rT < t < (r+1)T$ is given by the displaced function $f(t-rT)$ and hence the portion of the periodic function $f(t)$ in the interval $rT < t < (r+1)T$ is given by [result (40)]

$$f_{r+1}(t) = f(t-rT)\,\{H(t-rT) - H[t-(r+1)T]\} \quad \ldots \ldots \ldots \ldots (42a)$$

Thus the complete representation of the periodic function $f(t)$ will be represented as sum of the functions in (42a) for all interval in $0 < t < \infty$ i. e.

$$\boxed{f(t) = \sum_{r=0}^{\infty} f(t-rT)\,\{H[t-rT] - H[t-(r+1)T]\}} \quad \ldots \ldots \ldots (41)$$

Transforms using Heaviside unit step function :—

(1) Unit step function H (t) :—

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases}$$

$$L\{H(t)\} = \int_0^\infty e^{-st} H(t)\, dt$$

$$= \int_0^\infty e^{-st}\, dt = \frac{1}{s}$$

$$\therefore \quad \boxed{L\{H(t) = \frac{1}{s}} \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

Hence

$$\boxed{L^{-1}\left\{\frac{1}{s}\right\} = H(t)} \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

(II) **Displaced unit step function H (t — a) :—**

$$H(t-a) = 0, \quad t < a$$
$$= 1, \quad t \geqslant a$$

$$\therefore \quad L\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\, dt$$

$$= \int_a^\infty e^{-st}$$

$$= \frac{1}{s} e^{-as}$$

$$\therefore \quad \boxed{L\{H(t-a)\} = \frac{1}{s} e^{-as}} \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

Hence

$$\boxed{L^{-1}\left\{\frac{1}{s} e^{-as}\right\} = H(t-a)} \quad \ldots \quad \ldots \quad (45)$$

(III) The part of the displaced function f (t − a) for t ≥ a

The part of the curve of the function $f(t-a)$ beyond $t \geq a$ is given by,

$$f(t-a)\,H(t-a) = 0,\ t < a$$
$$= f(t-a),\ t \geq a$$

$$L\{f(t-a)\,H(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,H(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= e^{-as}\int_0^\infty e^{-su} f(u)\,du,\ [\,t = u + a\,]$$

$$= e^{-as}\bar{f}(s)$$

$$\therefore \quad \boxed{L\{f(t-a)\,H(t-a)\} = e^{-as}\bar{f}(s)} \quad \ldots \ldots \quad (46)$$

Hence

$$\boxed{L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)\,H(t-a)} \quad \ldots \ldots \quad (47)$$

Cor :— If $a = 0$

$$\boxed{L\{f(t)\,H(t)\} = \bar{f}(s)} \quad \ldots \ldots \ldots \ldots \quad (48)$$

Note :— In case, the Laplace transform of $f(t)\,H(t-a)$ is required, first express $f(t)$ as a function of $t - a$ i. e.

$$f(t) = \phi(t-a),$$

then

$$L\{f(t)\,H(t-a)\} = L\{\phi(t-a)\,H(t-a)\}$$
$$= e^{-as}\bar{\phi}(s).$$

This is illustrated in example 1 below.

(IV) The Periodic function $f(t) = f(t+T)$. The part $F(t)$ of the periodic function $f(t)$ in the interval $0 \leqslant t \leqslant T$ [*ref.* result (42)] is given by

$$F(t) = f(t)\,H(t) - f(t-T)\,H(t-T) \quad \dots \dots \quad \text{(i)}$$

i. e.

$$F(t) = f(t),\ 0 \leqslant t \leqslant T$$
$$= 0,\ t > T$$

Taking the Laplace transform of (i) and using the results (46) and (48), we get

$$F(s) = L\,[f(t)\,H(t) - f(t-T)\,H(t-T)]$$
$$= \bar{f}(s) - e^{-sT}\,\bar{f}(s)$$

Hence the Laplace transform of the periodic function $f(t)$ is given by

$$\boxed{\begin{aligned} \bar{f}(s) &= \frac{F(s)}{1-e^{-sT}} \\ \text{where } \bar{F}(s) &= \int_0^T e^{-su} f(u)\,du \end{aligned}} \quad \dots \dots \dots \quad (49)$$

Example 1 :—Find the Laplace transform of the function

(i) $[1+2t-3t^2+4t^3]\,H(t-2)$

(ii) $\sin t\;H(t-\pi)$

(i) Here $f(t) = 1+2t-3t^2+4t^3$. In order to find the Laplace transform $f(t)$ must be expressed as a function of $t-2$, so that we can apply the result (46)

Now $f(2) = 25, f'(2) = 38, f'''(2) = 42, f^{iv}(2) = 24$

$\therefore$ By Taylor's Theorem

$$f(t) = 25+38(t-2) + \frac{42}{2!}(t-2)^2 + \frac{24}{3!}(t-2)^3$$

$\therefore$ By results (46).

$$L\{f(t)\,H(t-2)\} = L\left\{\left[25+38(t-2) + \frac{42}{2!}(t-2)^2 + \frac{24}{3!}(t-2)^3\right]H(t-2)\right\}$$
$$= e^{-2s}\left[\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right]$$

(ii) Express $\sin t$ as a function of $t-\pi$, thus

$$\sin t = \sin (t-\pi+\pi)$$
$$= \sin (t-\pi) \cos \pi$$
$$= -\sin (t-\pi)$$

$$\therefore \quad L \sin t \, H(t-\pi) = -L\{\sin (t-\pi) \, H(t-\pi)\}$$
$$= \frac{e^{-\pi s}}{s^2+1}$$

Example 2 :— Express the following function in terms of Heaviside's unit step functions and hence find its Laplace transform

$$f(t) = \cos t. \; 0<t<\pi$$
$$= \sin t, \; t>\pi$$

For interval $0<t<\pi$

$$\cos t = (\cos t) \, H(t) - (\cos t) \, H(t-\pi)$$
$$= \cos t \, [H(t) - H(t-\pi)] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

and for $t>\pi$

$$\sin t = \sin t \, H(t-\pi) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Hence from (i) and (ii)

$$f(t) = \cos t \, [H(t) - H(t-\pi)] + \sin t \, [H(t-\pi)]$$
$$= \cos t \, H(t) + (\sin t - \cos t) \, H(t-\pi)$$

Using the results (48) and (46)

$$L\{\cos t . H(t)\} = \frac{s}{s^2+1} \quad \ldots \quad \ldots \quad (iii)$$

and

$$L\{(\sin t - \cos t) \, H(t-\pi)\}$$
$$= L\{[\sin (t-\pi+\pi) - \cos (t-\pi+\pi)] \, H(t-\pi)\}$$
$$= L\{[-\sin (t-\pi) + \cos (t-\pi)] \, H(t-\pi)\}$$
$$= \frac{e^{-\pi s}}{s^2+1} [s-1] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (iv)$$

Hence from (iv) and (iii), we have

$$L\{f(t)\} = \frac{1}{s^2+1}\left[s + (s-1) e^{-\pi s}\right]$$

Example 3 :— Express the full wave rectification of A sin pt [Fig. 65] in terms of unit step function and hence find the Laplace transform of this function

The part of the curve between $0< t < \frac{\pi}{p}$ is

$$f_1(t) = A \sin pt \left[H(t) - H\left(t - \frac{\pi}{p}\right)\right] \quad \ldots \quad \ldots \quad (i)$$

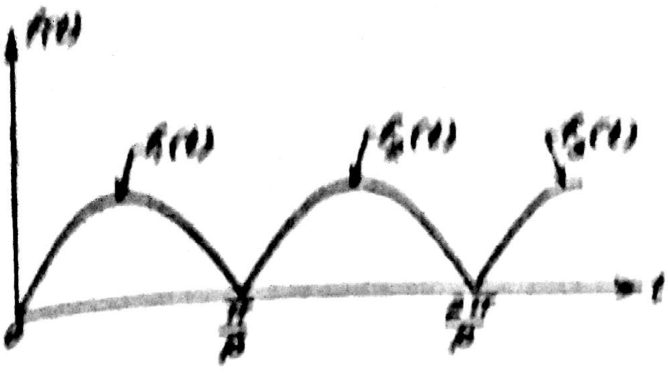


Fig. 65

The part of the curve between $\frac{\pi}{p} < t < \frac{2\pi}{p}$ is obtained by displacing the curve to the right through the distance $\frac{\pi}{p}$, hence the part of this displaced curve $A \sin p\left(t - \frac{\pi}{p}\right)$ in the interval $\frac{\pi}{p} < t < \frac{2\pi}{p}$ is given by

$$f_2(t) = A \sin p\left(t - \frac{\pi}{p}\right)\left[H\left(t - \frac{\pi}{p}\right) - H\left(t - \frac{2\pi}{p}\right)\right] \quad \ldots \quad \text{(ii)}$$

Thus in general the curve in the interval $(r-1)\frac{\pi}{p} < t < \frac{\pi r}{p}$ is obtained by displacing the curve $A \sin pt$, to the right through the distance $(r-1)\frac{\pi}{p}$ i. e. $\sin p\left[t - (r-1)\frac{\pi}{p}\right]$ and the part of this curve between $(r-1)\frac{\pi}{p} < t < \frac{\pi r}{p}$ is given by

$$f_r(t) = A \sin p\left[t - (r-1)\frac{\pi}{p}\right]\left\{H\left[t-(r-1)\frac{\pi}{p}\right] - H\left[t - \frac{r\pi}{p}\right]\right\} \quad \ldots \quad \ldots$$

Thus the full rectified wave [Fig. 65] is represented by the function

$$F(t) = \sum_{r=0}^{\infty} A \sin p\left(t - \frac{r\pi}{p}\right)\left\{H\left[t - \frac{r\pi}{p}\right] - H\left[t - \frac{(r+1)\pi}{p}\right]\right\} \quad \ldots \quad \text{(iv)}$$

Now by the result (46), we have

$$L\left\{A \sin p\left(t - \frac{r\pi}{p}\right) H\left(t - \frac{r\pi}{p}\right)\right\} = \frac{Ap}{s^2+p^2} e^{-\frac{r\pi}{p} s} \quad \ldots \quad \ldots \quad \text{(v)}$$

and

$$L\left\{A \sin p\left(t - \frac{r\pi}{p}\right) H\left[t - \frac{(r+1)\pi}{p}\right]\right\}$$

$$= L\left\{A \sin p\left[t - \left(\frac{r+1}{p}\right)\pi + \frac{\pi}{p}\right] H\left[t - \frac{(r+1)}{p}\pi\right]\right\}$$

[adjusted to use the result (47)]

$$= L\left\{- A \sin p\left[t - \frac{(r+1)\pi}{p}\right] H\left[t - \frac{(r+1)\pi}{p}\right]\right\}$$

$$= -\frac{Ap}{s^2 + p^2} \cdot e^{-\frac{(r+1)\pi s}{p}} \qquad \ldots \ldots \text{(vi)}$$

Hence taking the Laplace transform of eqt. (iv), we get from (v) and (vi),

$$\bar{F}(s) = \{ L f(t) \}$$

$$= \frac{Ap}{s^2 + p^2}\left(1 + e^{-\frac{\pi s}{p}}\right)\sum_{r=0}^{\infty} e^{-\frac{r\pi s}{p}}$$

$$= \frac{Ap}{s^2 + p^2}\left(\frac{1 + e^{-\pi s/p}}{1 - e^{-\pi s/p}}\right)$$

$$= \frac{Ap}{s^2 + p^2}\left[\frac{e^{\pi s/2p} + e^{-\pi s/2p}}{e^{\pi s/2p} - e^{-\pi s/2p}}\right] = \frac{Ap}{s^2 + p^2} \coth\left(\frac{\pi s}{2p}\right).$$

Aliter :—We can directly use the result (49). Here F (t) the function in the interval $0 < t < \frac{\pi}{p}$ is

$$F(t) = A \sin pt \qquad \left(T = \frac{\pi}{p}\right)$$

$$\therefore \quad F(s) = \int_0^{\pi/p} e^{-su} A \sin pu \, du$$

$$= A\left[-\frac{e^{-su}}{s^2 + p^2}(p \cos pu + s \sin pu)\right]_0^{\pi/p} = \frac{Ap}{s^2 + p^2}\left(1 + e^{-\pi s/p}\right)$$

By result (49), we get

$$\therefore \bar{f}(s) = \frac{Ap}{s^2 + p^2}\left(\frac{1 + e^{-\pi s/p}}{1 - e^{-\pi s/p}}\right) = \frac{Ap}{s^2 + p^2} \coth\left(\frac{\pi s}{2p}\right).$$

13.10. Dirac-delta Function (Unit Impulse Function) :—

Consider the function F (t) defined by

$$F(t) = 0, \quad t < a$$
$$\left.\begin{aligned} &= \frac{1}{\epsilon}, \quad a \leq t \leq a+\epsilon \\ &= 0, \quad t > a+\epsilon \end{aligned}\right\} \quad (50)$$

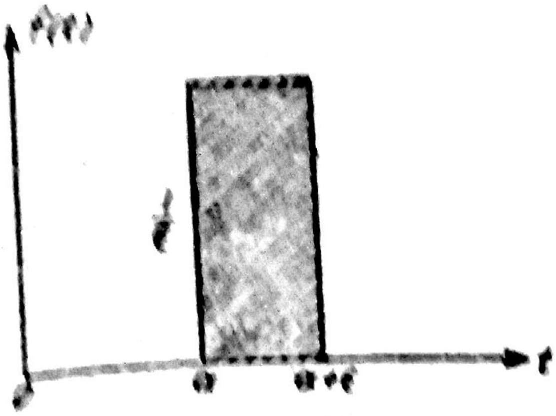


Fig. 66

This function is represented in the Fig. 66.

Then integrating the function F (t), we get

$$\int_0^\infty F(t)\, dt = \int_a^{a+\epsilon} \frac{1}{\epsilon}\, dt = 1 \text{ for all values of } \epsilon \quad \ldots \quad \ldots \quad (51)$$

As $\epsilon \to 0$ in the limit, the function F (t) tends to be *infinite* at $t = a$ and *zero* every where, with the characteristic property that its integral [resuls (51)] across $t = a$ is *unity*. If F (t) represents a force acting for short duration ϵ at time $t = a$, then the integral

$$\lim_{\epsilon \to 0} \int_a^{a+\epsilon} F(t)\, dt \quad (=1)$$

represents "*Unit impulse*" at $t = a$ [as the product of infinite force and infinitesimal time represents an impulse]. Hence limiting form of F (t) [as $\epsilon \to 0$] in (50) is expressed as "*Unit impulse function*" or "*Dirac-delta function*" denoted by $\delta(t-a)$ i. e,

$$\boxed{\delta(t-a) = \lim_{\epsilon \to 0} F(t)} \quad \ldots \quad \ldots \quad \ldots \quad (53)$$

where F (t) is given by (50).

When $a = 0$, the *unit impulse function* at $t = 0$ is given by

$$\boxed{\delta(t) = \lim_{\epsilon \to 0} F(t)} \quad \ldots \quad \ldots \quad \ldots \quad (54)$$

where
$$\left.\begin{aligned} F(t) &= 0,\ t < 0 \\ &= \frac{1}{\varepsilon},\ 0 \leqslant t \leqslant \varepsilon \\ &= 0,\ t > \varepsilon \end{aligned}\right\} \qquad \ldots \ \ldots \ \ldots \qquad (55)$$

Laplace transform of Dirac-delta function :— From the definition of $F(t)$ given in (50), the Laplace transform of $F(t)$ is given by

$$\bar{F}(s) = L\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt$$

$$= \frac{1}{\varepsilon} \int_0^{a+\varepsilon} e^{-st} dt$$

$$= \frac{e^{-as}}{s} \left[\frac{1 - e^{-\varepsilon s}}{\varepsilon} \right]$$

$$\therefore \quad L\{\delta(t-a)\} = \lim_{\varepsilon \to 0} \bar{F}(s)$$

$$= \frac{e^{-as}}{s} \lim_{\varepsilon \to 0} \left[\frac{1 - e^{-\varepsilon s}}{\varepsilon} \right]$$

$$= e^{-as} \text{ [by L' Hopital rule]}$$

$$\boxed{\therefore L\{\delta(t-a)\} = e^{-as}} \qquad \ldots \ \ldots \qquad (55)$$

Cor :— when $a = 0$

$$\boxed{L\{\delta(t)\} = 1} \qquad \ldots \ \ldots \qquad (56)$$

From the definition of Dirac-delta function, we can now represent the following physical problem, in terms of this function.

(a) Concentrated load W, applied to the beam at $x = a$, is represented as

$$W\,\delta(x-a) \qquad \ldots \ \ldots \ \ldots \ \ldots \qquad (57)$$

(b) The unit couple C applied at $x = a$ is a limiting form of the couple formed by two unit forces acting at $x=a$ and $x = a + \delta x$ in opposite directions, whan $\delta x \to 0$ i. e. the couple C acting at $x = a$ is given by

$$C(x-a) = \lim_{\delta x \to 0} \frac{\delta(x-a) - \delta(x-a-\delta x)}{\delta x}$$

$$\therefore \quad L\{C(x-a)\} = \lim_{\delta x \to 0} \frac{1}{\delta x}\left\{e^{-as} - e^{-(a+\delta x)s}\right\}$$

[by result (56)

$$= se^{-as}$$

(c) A force of impulse J applied at time $t = a$ is given by

$$J\,\delta(t-a)$$

Relation between Heaviside unit step function and Dirac-dalta function :—From the definition of F (t) as given by the result (51) and from the Fig. 66, the function F(t) in terms of Heaviside function is

$$F(t) = \frac{1}{\epsilon}[H(t-a) - H(t-a-\epsilon)] \quad \ldots \quad \ldots \quad (58)$$

$$\therefore \quad \delta(t-a) = \lim_{\epsilon \to 0} F(t)$$

$$= \lim_{\epsilon \to 0} \frac{H(t-a) - H(t-a-\epsilon)}{\epsilon}$$

$$= \frac{d}{dt} H(t-a) = H'(t-a)$$

i. e. $$\boxed{\delta(t-a) = H'(t-a)} \quad \ldots \ldots \ldots \ldots \ldots \quad (59)$$

Properties of Dirac-delta function :—By definition of the unit step function H $(t-a)$ and $\delta(t-a)$, we have

$$H(0) \neq 0,\ H(0) = \delta(0) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

(A) $L\,H'(t-a) = L\,\delta(t-a)$

This is evident from the result (59) or using the result

$$L\{f'(t)\} = s\bar{f}(s) - f(0),$$

we have

$$L\ H'(t-a) = s\,\bar{H}(s) - H(0)$$

$$= s \cdot \frac{1}{s} e^{-as} \quad \text{[by result (44) and (i)]}$$

$$= e^{-as} = L\,\delta(t-a) \quad \text{[by result (55)}$$

(B) $L\,H''(t-a) = L\,\delta'(t-a) = se^{-as}$

since

$$L\left\{ f''(t) \right\} = s^2 \bar{f}(s) - s f(0) - f'(0)$$

we get

$$L\,H''(t-a) = s^2\,\bar{H}(s) - sH(0) - H'(0)$$

$$= s^2 \cdot \frac{1}{s} \cdot e^{-as} \quad \text{[from (i)]}$$

$$= se^{-as} = L\,\delta'(t-a).$$

(C) **Shifting property** :—

Since,

$$L\,\delta(t-a) = e^{-as}$$

we have

$$\int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-as}$$

i. e. the value of the integration of $e^{-st}\,\delta(t-a)$ in the limits 0 to ∞ is given by replacing t in e^{-st} the coefficient of $\delta(t-a)$, by a i. e. e^{-as}

This shifting property can be generalised as,

$$\boxed{\int_0^\infty f(t)\,\delta(t-a) = f(a)} \qquad (60)$$

For

$$\delta(t-a) = \lim_{\epsilon \to 0} F(t)$$

where F (t) is given by (50) i. e.,

$$F(t) = \frac{1}{\epsilon}\left[H(t-a) - H(t-a-\epsilon) \right]$$

$$\therefore \int_0^\infty f(t)\,\delta(t-a)\,dt = \lim_{\epsilon \to 0} \int_0^\infty \frac{1}{\epsilon} f(t)\left[H(t-a) - H(t-a-\epsilon) \right] dt$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int_a^{a+\epsilon} f(t)\,dt \right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\epsilon f(\xi) \right], \text{ where } a < \xi < a+\epsilon$$

[By mean value Th. of integral]

$$= \lim_{\epsilon \to 0} \left[f(\xi) \right]$$

$$= f(a) \qquad [\text{ as } \xi \to a \text{ as } \epsilon \to 0]$$

Solved Problems :—

Example 1 :— Obtain the inverse transforms of

(i) $\dfrac{e^{-s}}{(s+1)^3}$ (ii) $\dfrac{e^{-\pi s}}{s^2+4}$

(i) since $L^{-1}\dfrac{1}{(s+1)^3} = \dfrac{t^2}{2!}$, by result (48), we get

$$L^{-1}\left\{ \frac{e^{-s}}{(s+1)^3} \right\} = \frac{(t-1)^2}{2!} H(t-1) \; [\text{as } a = 1]$$

$$\text{i. e. } f(t) = \frac{(t-1)^2}{2!} \text{ for } t > 1$$

$$= 0 \text{ for } 0 < t < 1$$

(ii) since $L^{-1}\dfrac{1}{s^2+4} = \frac{1}{2}\sin 2t$, by result (47)

$$L^{-1}\left\{ \frac{e^{-\pi s}}{s^2+4} \right\} = \tfrac{1}{2}\sin 2(t-\pi)\, H(t-\pi) \quad [\text{as } a = \pi]$$

$$= \tfrac{1}{2}\sin 2t\, H(t-\pi)$$

$$\text{i. e. } f(t) = 0,\; 0 < t < \pi$$

$$= \sin 2t,\; t > \pi$$

Example 2 :– Solve the differential equation ($n \neq 1$).

$$\frac{d^2y}{dt^2} + n^2y = f(t), \text{ with } y(0) = y'(0) = 0$$

where

$$\begin{aligned} f(t) &= 0,\ 0 < t < \pi \\ &= \sin t,\ \pi < t < 2\pi \\ &= 0,\ t > 2\pi \end{aligned}$$

Expressing $f(t)$ in terms of unit step function, we get

$$f(t) = \sin t\,[H(t-\pi) - H(t-2\pi)]$$

Thus the differential equation is

$$\begin{aligned} \frac{d^2y}{dt^2} + n^2y &= \sin t\,[H(t-\pi) - H(t-2\pi)] \\ &= \sin(t-\pi+\pi)\,H(t-\pi) \\ &\qquad - \sin(t-2\pi+2\pi)\,H(t-2\pi) \\ &= -\sin(t-\pi)\,H(t-\pi) - \sin(t-2\pi)\,H(t-2\pi) \end{aligned}$$

Taking the Laplace transform, we get

$$[s^2\bar{y} - s\,y(0) - y'(0)] + n^2\bar{y} = \frac{-1}{s^2+1}\left[e^{-\pi s} + e^{-2\pi s}\right]$$

[by result (46)]

$$\therefore \quad \bar{y} = \frac{-1}{(s^2+1)(s^2+n^2)}\left[e^{-\pi s} + e^{-2\pi s}\right] \quad \ldots \ldots \ldots \text{(i)}$$

[as $y(0) = y'(0) = 0$]

For inverse transform, we have

$$\frac{1}{(s^2+1)(s^2+n^2)} = \frac{1}{1-n^2}\left[\frac{1}{s^2+n^2} - \frac{1}{s^2+1}\right]$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s^2+1)(s^2+n^2)}\right\} = \frac{1}{n^2-1}\left\{\frac{1}{n}\sin nt - \sin t\right\}$$

Hence taking inverse transform by using the result (47), we have from (i)

$$\begin{aligned} y = \frac{1}{n^2-1}\Big\{&\left[\frac{1}{n}\sin n(t-\pi) - \sin(t-\pi)\right]H(t-\pi) \\ &+ \left[\frac{1}{n}\sin n(t-2\pi) - \sin(t-2\pi)\right]H(t-2\pi)\Big\} \\ = \frac{1}{n^2-1}\Big\{&\left(\frac{1}{n}\cos n\pi \sin nt + \sin t\right)H(t-\pi) \\ &+ \left(\frac{1}{n}\sin nt - \sin t\right)H(t-2\pi)\Big\} \end{aligned}$$

when $0 < t < \pi$, $\quad H(t-\pi) = H(t-2\pi) = 0$

$\pi < t < 2\pi$, $\quad H(t-\pi) = 1,\ H(t-2\pi) = 0$

$t > 2\pi$, $\quad H(t-\pi) = H(t-2\pi) = 1$

Hence the solution of the eqt. is

$$y = 0, \quad 0 < t < \pi$$

$$= \frac{1}{n(n^2-1)}\left(\cos n\pi \sin nt + n \sin t\right), \quad \pi < t < 2\pi$$

$$= \frac{1}{n(n^2-1)}\left[(1 + \cos n\pi)\sin nt\right], \quad t > 2\pi.$$

For problems on beam-deflections when a concentrated load is applied to a beam or when it is partially loaded, we use the differential equation

$$EI\,\frac{d^4y}{dx^4} = w(x) \qquad \ldots \quad (61)$$

where $w(x)$ is transverse load intensity at a distance x from the origin and the end conditions are given by,

(a) Clamped ends :— $y = \dfrac{dy}{dx} = 0$ (slopes are zero)

(b) Supported end :— $y = \dfrac{d^2y}{dx^2} = 0$ (Bending moment zero)

(c) Free end :— $\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = 0$ (Bending moment and shearing stress are zero)

Example 3 :—Obtain the deflection of weightless beam of length l and freely supported at ends, when a concentrated load W acts at $x = a$.

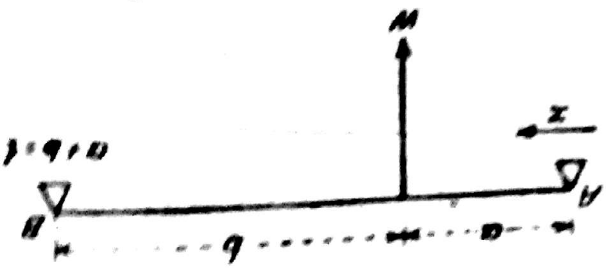


Fig. 67.

The differential equation of the beam is

$$EI\,\frac{d^4y}{dx^4} = W\,\delta(x-a) \qquad \ldots \quad (i)$$

with the end conditions

$$y(0) = y''(0) = 0 \qquad \ldots \quad (ii)$$

$$y(l) = y''(l) = 0 \qquad \ldots \quad (iii)$$

and assume

$$y'(0) = A, \; y'''(0) = B \qquad \ldots \quad (iv)$$

Taking the Laplace transform of eqt. (i), we get

$$\left[s^4\bar{y} - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0)\right] = \frac{W}{EI}\,e^{-as}$$

[from result (56)]

i.e From (ii) and (iv), we get

$$s^4 y = As^2+B+\frac{W}{EI}e^{-as}$$

$$\therefore\ y=\frac{A}{s^2}+\frac{B}{s^4}+\frac{W}{EI}\cdot\frac{e^{-as}}{s^4}$$

$\therefore$ Taking inverse transform [using the result (47)],

$$y=Ax+B\frac{x^3}{3!}+\frac{W}{EI}\frac{(x-a)^3}{3!}\ H(x-a) \quad \ldots \quad \ldots$$

$$\therefore\ y'=A+\tfrac{1}{2}.B\,x^2+\frac{W}{2EI}(x-a)^2\,H(x-a)$$

$$y''=Bx+\frac{W}{EI}(x-a)\,H(x-a)$$

Now use the conditions (iii), we get

$$y(l)=0=Al+B\frac{l^3}{6}+\frac{W}{EI}\cdot\frac{b^3}{6},\ [\ l-a=b\ ]$$

$$y''(l)=0=Bl+\frac{W}{EI}\,b$$

[as H $(x-a)=1$, for $x>$

$$\therefore\ B=-\frac{W}{EI}\ \frac{b}{l}$$

$$\text{and } A=-\frac{1}{l}\left[B\frac{l^3}{6}+\frac{W}{EI}\ \frac{b^3}{6}\right]$$

$$=-\frac{W}{6EIl}[-bl^3+b^3]=\frac{W}{6EIl}\,b\,(l^2-b^2)$$

$$=\frac{W}{6EIl}\,b\,(l+b)\,.\,(l-b)=\frac{W}{6EI}\ \frac{ab\,(l+b)}{l}$$

Hence from (v), the solution is

$$EI\,y=\frac{W}{6}\left\{\frac{ab\,(l+b)}{l}x-\frac{b}{l}\ x^3+(x-a)^3\,H(x-a)\right\}$$

i. e. when $0<x<a$, H $(x-a)=0$

$$\therefore\ EI\ y=\frac{W}{6}\left\{\frac{ab\,(l+b)}{l}x-\frac{b}{l}\,x^3\right\} \quad \ldots \quad \ldots \quad \ldots$$

for $a<x<l$, H $(x-a)=1$

$$EI\,y=\frac{W}{6}\left\{\frac{ab\,(l+b)}{l}x-\frac{b}{l}\ x^3+(x-a)^3\right\} \quad \ldots \quad \ldots$$

The deflection of the beam at the load W is [from (vi) or (vii)]

$$y(a)=\frac{W}{6\,EI\,l}\ [a^2b\,(l+b)-ba^3]$$

$$= \frac{W}{6\,EI\,l}\, a^2 b\,[\,(l+b) - a\,]$$

$$= \frac{W}{6\,EI\,l}\, a^2 b\,.\,(2b) = \frac{1}{3}\frac{W}{EI}\,.\,\frac{a^2 b^2}{l}.$$

Example 4 : – A cantilever beam of uniform weight W per unit length and length l is fixed at the end $x=0$ and has uniformly distributed load P (constant) per unit length in $\frac{l}{2} < x < l$. Find the deflection curve of the beam and the deflection at the free end.

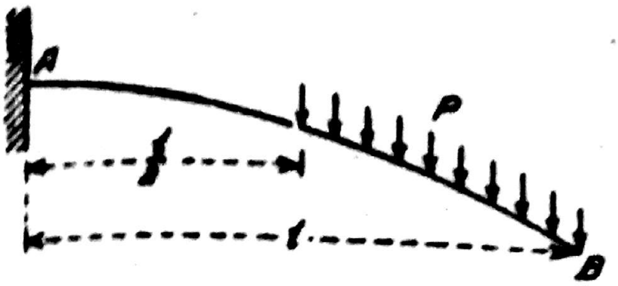


Fig. 68

The differential equation of the beam is

$$EI\,\frac{d^4y}{dx^4} = f(x) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (i)$$

where $f(x)$ represents load intensity per unit length at arbitrary point x. $f(x)$ for the problem is given by

$$f(x) = W,\; 0 < x < \frac{l}{2}$$

$$= W + P,\; \frac{l}{2} < x < l.$$

Thus in terms of Heaviside unit step function,

$$f(x) = W + PH\left(x - \frac{l}{2}\right)$$

Hence the differential equation (i) becomes,

$$EI\,\frac{d^4y}{dx^4} = W + PH\left(x - \frac{l}{2}\right) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (ii)$$

with end conditions,

$$x = 0,\; y'(0) = y(0) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (iii)$$

$$x = l,\; y'(l) = y''(l) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (iv)$$

Taking Laplace transform of (ii), we get

$$s^4\bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{1}{El}\left[\frac{W}{s} + \frac{P}{s}\,e^{-\frac{ls}{2}}\right]$$

Hence from conditions (iii),

$$\bar{y} = \frac{A}{s^3} + \frac{B}{s^4} + \frac{1}{EI}\left(\frac{W}{s^5} + \frac{P}{s^5}\,e^{-\frac{ls}{2}}\right) \quad \ldots \ldots \ldots \qquad (v)$$

$[y'(0) = A,\; y''(0) = B]$

Taking inverse transform of (v), we have

$$y(x) = A\frac{x^2}{2} + B\frac{x^3}{6} + \frac{1}{24\,EI}\left[Wx^4 + P\left(x - \frac{l}{2}\right)^4 H\left(x - \frac{l}{2}\right)\right] \quad \text{(vi)}$$

$$y'(x) = Ax + B\frac{x^2}{2} + \frac{1}{6EI}\left[Wx^3 + P\left(x - \frac{l}{2}\right)^3 H\left(x - \frac{l}{2}\right)\right]$$

$$y''(x) = A + Bx + \frac{1}{2EI}\left[Wx^2 + P\left(x - \frac{l}{2}\right)^2 H\left(x - \frac{l}{2}\right)\right]$$

$$y'''(x) = B + \frac{1}{EI}\left[Wx + P\left(x - \frac{l}{2}\right) H\left(x - \frac{l}{2}\right)\right]$$

From conditions (iv), we have as $H\left(x - \frac{l}{2}\right) = 1$ for $x > \frac{l}{2}$

$$0 = A + Bl + \frac{1}{2EI}\left[Wl^2 + \frac{Pl^2}{4}\right]$$

$$0 = B + \frac{1}{EI}\left[Wl + \frac{Pl}{2}\right]$$

i. e. $B = -\frac{l}{2EI}(2W + P)$ and $A = \frac{l^2}{8\,EI}(4W + 3P)$.

Hence from (vi) the deflection curve is

$$y(x) = \frac{l^2}{16\,EI}(4W + 3P)\,x^2 - \frac{l}{12EI}(2W + P)\,x^3 + \frac{Wx^4}{24EI} + \frac{P}{24\,EI}\left(x - \frac{l}{2}\right)^4 H\left(x - \frac{l}{2}\right).$$

Thus deflection at free end $x = l$ is given by

$$\delta = y(l) = \frac{l^4}{384EI}(48\,W + 41P)$$

Examples XIII-D

1. Express the following functions in terms of *Heaviside unit step function* and hence find their Laplace transforms :—

(i) $f(t) = (t - a)^4,\quad t > a$
$\qquad = 0,\quad 0 < t < a$

(ii) $f(t) = e^t \cos t,\quad 0 < t < \pi$
$\qquad = e^t \sin t,\quad t > \pi$

(iii) $f(t) = e^{-t}\quad 0 < t < 3$
$\qquad = 0,\quad t > 3$

(iv) $f(t) = t^2,\quad 0 < t < 1$
$\qquad = 4t,\quad t > 1$

(v) $f(t) = \cos t,\quad 0 > t < \pi$
$\qquad = \cos 2t,\quad \pi < t < 2\pi$
$\qquad = \cos 3t.\quad t > 2\pi$

2. Find Laplace transforms of

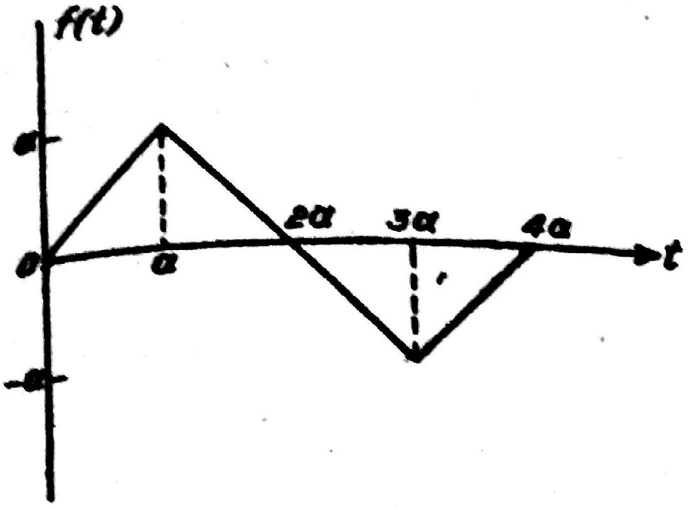
(i) $t\,H(t-4) - t^2\,\delta(t-4)$

[use result (60)

(ii) $(\sin 2t)\,\delta(t-1)$

(iii) $t^2\,H(t-1)$

(iv) $e^{4t}\,\text{erf}(\sqrt{t})$



Fig, 69

3. Find the Laplace transform of the function given by the graph of Fig. 69.

4. Find the Laplace transforms of the following periodic functions.

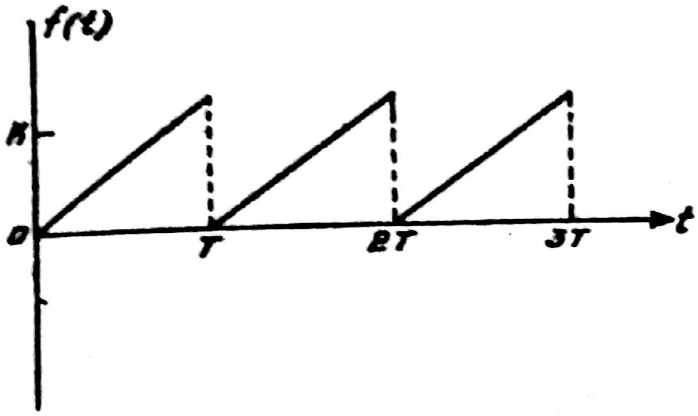


Fig. 70.

(i) Saw toothed wave function (Fig. 70)

$$f(t) = \frac{kt}{T},\ 0 < t < T$$

and $f(t) = f(t+T)$

(ii) Half wave rectified sine wave (Fig. 71)

$$j(t) = a \sin pt,\ 0 < t < \frac{\pi}{p}$$

$$= 0\ .\ \frac{\pi}{p} < t < \frac{2\pi}{p}$$

and $f(t) = f\left(t + \dfrac{2\pi}{p}\right)$

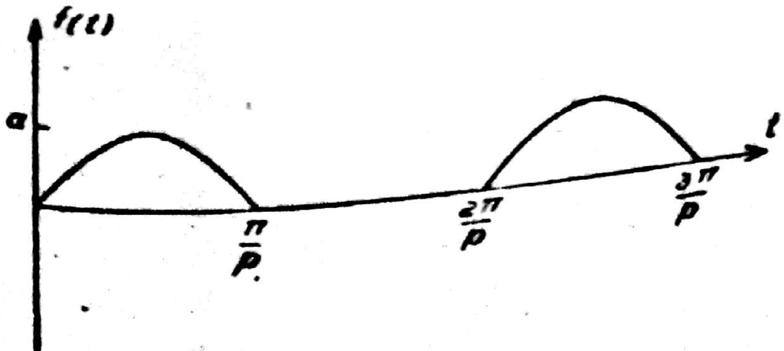


Fig. 71

(iii) **Rectified triangular wave function (Fig. 72)**

$$f(t) = \frac{t}{a},\ 0 < t < a$$

$$= \frac{1}{a}(2a - t),\ a < t < 2a$$

and $f(t) = f(t+2a)$.

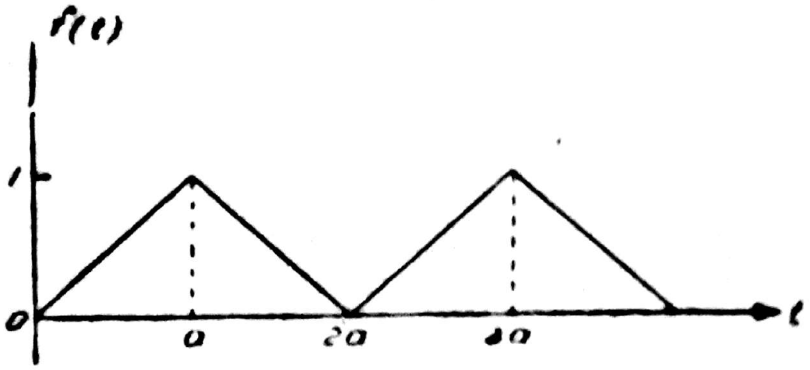


Fig. 72.

(iv) **Stair-Case wave function (Fig. 73)**

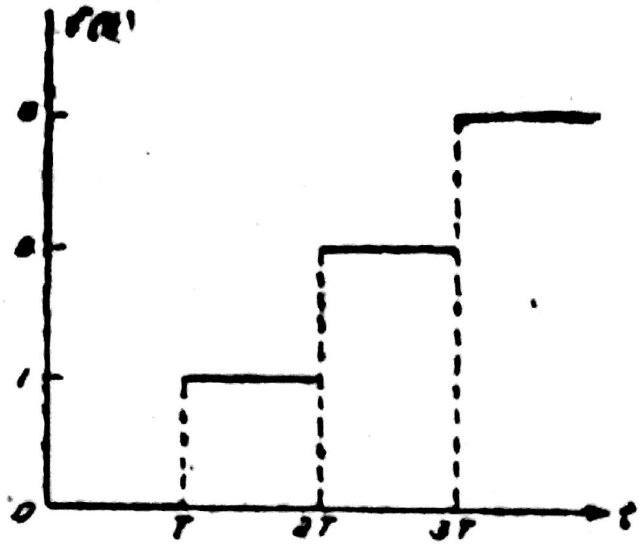


Fig. 73

$y = r - 1,\ (r-1)\,T < t < rT$

(v) $f(t) = t^2,\quad 0 < t < 2$

and $f(t) = f(t+2)$

(vi) $f(t) = \left.\begin{aligned} &t,\ 0 < t < 1 \\ &0,\ 1 < t < 2 \end{aligned}\right\}$

and $f(t) = f(t+2)$

5. **Bvaluate**

(i) $\displaystyle\int_{-\infty}^{\infty} e^{-t}\,H(t-a)\,dt$ (ii) $\displaystyle\int_{0}^{\infty} (\cos 2t)\,\delta\left(t - \frac{\pi}{3}\right)dt$

6 **Solve the following differential equations :—**

(i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10\,y = f(x),\ y(0) = y'(0) = 0$

where $\left.\begin{aligned} f(x) &= 1,\quad 0 < x < \pi \\ &= -1,\ \pi < x < 2\pi \end{aligned}\right\}$ and $f(x) = f(x+2\pi)$

(ii) $\frac{d^2y}{dt^2} + y = H(t-\pi) - H(t-2\pi), y(0) = y'(0) = 0$

(iii) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x), y(0) = y'(0) = 0$

where $f(x) = 1, \quad 0 < x < 1$

$= 0, \quad x > 1.$

7. A particle of mass m moves in a st. line, under a restoring force mn^2x. At $t = 0$, $x = \dot{x} = 0$ and a constant force P acts for a time T from $t = 0$ Show that

$$x = \frac{P}{n^2}[1 - H(t-T) - \frac{P}{n^2}[\cos nt - H(t-T)\cos n(t-T)]$$

8. A mass is suspended from a spring of stiffness $\lambda^2 m$ and is set in motion from equilibrium position at $t = 0$, by a blow of impulse J. Show that the displacement, at time t, is given by

$$x = \frac{J}{\lambda m}\sin \lambda t.$$

9. Find the steady current in the circuit containing L, R in series, when the applied e. m. f. is given by $E(t) = t, 0 < t < 1$.
and $E(t+1) = E(t)$.

10. An e. m. f. E_0 is applied to LCR circuit in series with zero initial conditions, between $t = 0$ and $t = 1$. Show that

$$i = \frac{E_0}{Ln}\{H(t)\, e^{-\mu t}\sin nt - H(t-1)\, e^{-\mu(t-1)}\sin n(t-1)\},$$

where $\mu = \frac{R}{2L}, n^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$.

11. In the problem 10, show that the current, when impulsive voltage E_0 is applied to the circuit at $t = 0$, is

$$i = \left(\frac{E_0}{Ln}\right)e^{-\mu t}\left\{n\cos nt - \mu\sin nt\right\}.$$

12. A constant voltage E_0 is applied between $t = 0$ and $t = T$ to the CR circuit in series, with zero initial conditions. Show that

$$i = \left(\frac{E_0}{R}\right)\left\{e^{-t/RC}H(t) - e^{-(t-T)/RC}H(t-T)\right\}$$

13. A light beam of length l is fixed at two ends and a concentrated load P acts at $x = a$. Show that

$$EIy = \frac{Pab^2}{l^2}\cdot\frac{x^2}{2} - \frac{Pb^2}{l^3}(3a+b)\frac{x^3}{6} + \frac{P(x-a)^3}{6}H(x-a)$$

[where $b = l -$

and the deflection at $x = a$, is given by

$$\frac{1}{3}\cdot\frac{P}{EI}\frac{a^3b^3}{l^3}.$$

14. A light beam of length l is fixed at $x = 0$ and free at $x = l$. If the concentrated load P acts $x = a$, show that

$$EIy = \frac{Px^2}{6}(3a - x), \quad 0 < x < a$$
$$= \frac{Pa^2}{6}(3x - a), \quad a < x < l.$$

15. A light beam of length l is hinged at $x = 0$ and $x = l$ and concentrated loads P_1 and P_2 act at $x = \frac{l}{3}$ and $x = \frac{2l}{3}$. Find the deflection.

16. A light beam of length l has its ends $x = 0$ and $x = l$ fixed and carries a load $P(x)$ per unit length given by

$$P(x) = 0, \quad 0 < x < \frac{l}{2}$$
$$= \lambda x, \quad \frac{l}{2} < x < l \quad (\lambda = \text{const})$$

and a concentrated load P_0 acts at $x = \frac{l}{3}$. Find the deflection

17. A light beam of length l has its ends $x = 0$ and $x = l$ hinged. If concentrated load W_0 acts at the point $x = \frac{l}{3}$, show that the deflection is given by

$$EIy = \frac{W_0}{81} x(5l^2 - 9x^2) + \frac{W_0}{6}\left(x - \frac{l}{3}\right)^3 H\left(x - \frac{l}{3}\right)$$

Answers

(1) (i) $(t-a)^4 H(t-a)$, $\frac{24}{s^5} e^{-as}$

(ii) $e^t \cos t \cdot H(t) + e^t(\sin t - \cos t) H(t-\pi)$,
$$\frac{(s-1) - (s-2)e^{-\pi s}}{s^2 - 2s + 2}$$

(iii) $e^{-t}[H(t) - H(t-3)]$, $\frac{1 + e^{-3s}}{s+1}$

(iv) $t^2 H(t) + (4t - t^2) H(t-1)$, $\frac{2 + (3s^2 + 2s - 2)e^{-s}}{s^3}$

(v) $\cos t \cdot H(t) + (\cos 2t - \cos t) H(t-\pi) + (\cos 3t) H(t - 2\pi)$
$$\frac{s}{s^2+1} + \left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right]e^{-\pi s} - \frac{s}{s^2+9}e^{-2\pi s}$$

(2) (i) $\frac{e^{-4s}}{s^3}(16s^2 + 4s + 1)$ (ii) $e^{-2s} \sin 4$

(iii) $e^{-s}\left[\frac{1}{s} + \frac{3}{s^2} + \frac{6}{s^3} + \frac{6}{s^4}\right]$ (iv) $\frac{1}{(s-4)\sqrt{s-3}}$

(3) $\frac{1}{s^2}\left[1+2e^{-as}+2e^{-3as}-e^{-4as}\right]$

(4) (i) $k\left[\frac{1}{Ts^2}-\frac{e^{-sT}}{s(1-e^{-Ts})}\right]$ (ii) $\frac{ap}{(s^2+p^2)(1-e^{-\pi s/p})}$

(iii) $\frac{1}{as^2}\tanh\frac{as}{2}$

(iv) $\frac{1}{s}\left\{1+e^{-sT}+e^{-2sT}+\dots\dots\dots\right\}$

(v) $\frac{2-2(1+2s+2s^2)e^{-2s}}{s^3(1-e^{-2s})}$ (vi) $\frac{1-(s+1)e^{-s}}{s^2(1-e^{-2s})}$

(5) (i) e^{-a} (ii) $-\frac{1}{2}$.

(6) (i) $\frac{1}{5}\sum_{r=0}^{\infty}(-1)^r\left[1-f(t-r\pi)\right]H(t-r\pi)$,

$f(t)=e^{-t}\left[\cos 3t+\frac{1}{3}\sin 3t\right]$

(ii) $y=(1+\cos t)\,H(t-\pi)-(1-\cos t)\,H(t-2\pi)$

or

$$=\begin{cases}0, & 0<t<\pi\\ 1+\cos t, & \pi<t<2\pi\\ 2\cos t, & t>2\pi\end{cases}$$

(iii) $y=\left[\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}\right]$

$-\left[\frac{1}{2}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)}\right]H(t-1)$

(9) $i=\sum_{r=0}^{\infty}\left\{\frac{1}{k}\left[1-e^{-k(t-r\pi)}\right]H(t-r\pi)+\right.$

$\left.\left[\left(\pi+\frac{1}{k}\right)e^{-k[t-(r+1)\pi]}-\frac{1}{k}\right]H[t-(r+1)\pi]\right\}$.

(15) $EI\,y(x)=\frac{l^2}{81}(5P_1+4P_2)\,x-\frac{1}{18}(2P_1+P_2)\,x^3$

$+\frac{1}{6}P_1\left(x-\frac{l}{3}\right)^3H\left(x-\frac{l}{3}\right)+\frac{1}{6}P_2\left(x-\frac{2l}{3}\right)^3H\left(x-\frac{2l}{3}\right)$

(16) $EI\,y(x)=\frac{13\lambda l^3}{1280}x^2-\frac{\lambda l^2}{192}x^3+\lambda\left[\left\{\frac{l}{48}(x-l/2)^4\right.\right.$

$\left.+\frac{1}{60}\left(x-\frac{l}{2}\right)^5\right\}H\left(x-\frac{l}{2}\right)$

$\left.-\left\{\frac{l}{24}(x-l)^4+\frac{1}{60}(x-l)^5\right\}H(x-l)\right]$.

CHAPTER 13

CURVE TRACING AND STANDARD CURVES

For evolution of areas, volumes of revolution and many other properties, it is essential to know the general form of the curve represented by an equation. Hence in this chapter, the general problem of curve tracing is discussed. Moreover the properties of some of the important curves which frequently occur in engineering applications, are dealt here.

13·1 Cartesian curves :—

The following are some of the main rules which will help to obtain the general form of curves from their equations.

(A) If the powers of y in the equation are even, the curve has a symmetry about x-axis.

(B) If the powers of x in the equation are even, the curve has a symmetry about y-axis.

(C) If the interchange of x and y, leaves the equation unaltered, then the curve is symmetrical about the line $y = x$.

(D) Find if origin is on the curve. If it is, find the tangents at the origin by equating to zero the lowest degree terms.

(E) Find the points of intersection with the co-ordinate axes and if necessary get the directions of the tangents at these points.

(F) If possible express the equation in the explicit form say $y = f(x)$ and examine how y varies as x varies continuously and also note when x or y is imaginary which give the regions where the part of the curve does not exist.

(G) Find the asymptotes and the position of the curve with respect to them. To get equation of asymptotes parallel to x-axis (or y-axis) equate to zero the cofficient of highest degree terms in x (or y). e. g. For the curve $x^4 + x^2y^2 - a^2(a^2 + y^2) = 0$ asympotes parallel to $y - ax$ is are $x^2 - a^2 = 0$ or $x = \pm a$.

To get oblique asympotes : Let $y = mx + c$ be the asymptote. The point of intersection with the curve $f(x\ y) = 0$ are given by $f(x, mx+c) = 0$. Equate to zero the coefficients of two highest powers of x, giving equations to determine m and c for example equation of aympote for the curve $x^3 + y^3 = 3axy$ is $x + y + a = 0$.

(H) If necessary transform the equation to the polar co-ordinates.

Examples :—

Type $y^2 = f(x)$:

(1) *Trace the curve $ay^2 = x^2(a - x)$.*

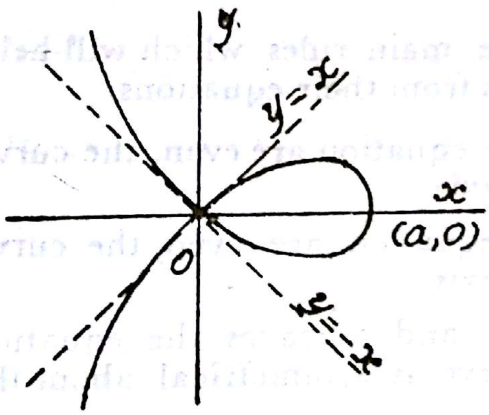


Fig. 107

We note :—

(i) There is symmetry about x-axis

(ii) Origin is a point on the curve and the tangents at the origin are given by

$a(y^2 - x^2) = 0$. i.e. $y = \pm x$.

(iii) The curve meets the x-axis at the points $(0, 0)$ and $(a, 0)$.

(iv) When $x > a$, y^2 is negative and thus curve does not exist beyond $x = a$. For $x < 0$, y^2 is positive and increases as x increases in the negative direction.

The form of the curve is given in the Fig. 107.

(2) *Trace the curve $y^2(x + a) = x^2(3a - x)$.*

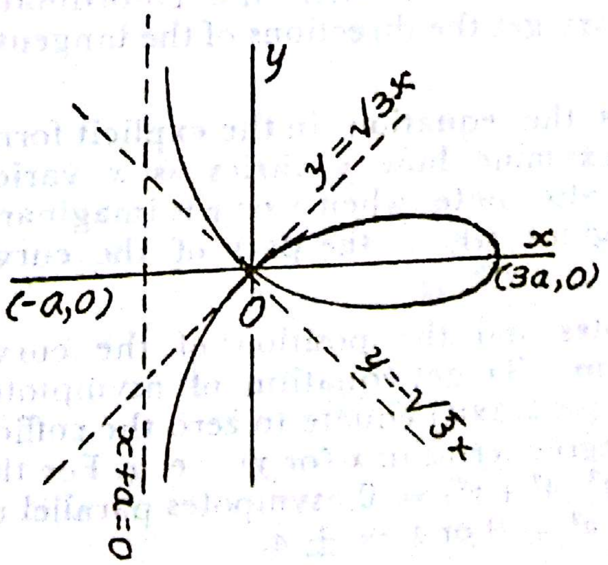


Fig. 108

(i) Curve is symmetrical about x axis.

(ii) Origin is a point on the curve and the tangents at the origin are given by

$y = \pm\sqrt{3}\,x$

(iii) The curve meets the x axis at points $(0, 0)$ and $(3a, 0)$

and $\frac{dy}{dx} = \infty$ at $x = 3a$.

(iv) Where $x = -a$, y is infinite, hence, $x + a = 0$ is the asymptote.

(v) When $x > 3a$ and $x < -a$, y^2 is negative and hence curve does not exist for the region where $x > 3a$ and $x < -a$.

The form on the curve is as shown in the Fig. 108.

(3) *Trace the curve* $a^2x^2 = y^3(2a - y)$.

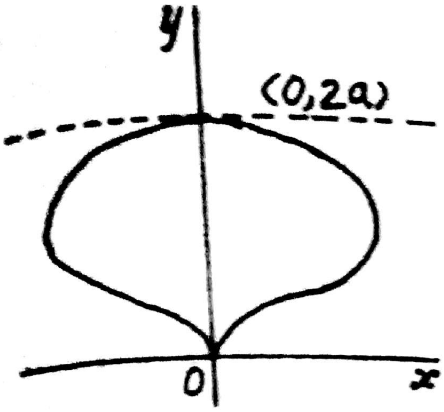


Fig. 109

(i) The curve has a symmetry about y axis.

(ii) It passes through the origin and $x = 0$ is the tangent at the origin.

(iii) It meets the y axis at the points $(0, 0)$ and $(0, 2a)$ and as $\frac{dy}{dx} = 0$ at the point $(0, 2a)$, the tangent at this point is parallel to x-axis.

(iv) For values of $y > 2a$ and $y < 0$, the curve does not exist as x^2 is negative for these points.

Fig. 109 gives the form of the curve.

(4) *Trace the curve* $y^2(a^2 + x^2) = x^2(a^2 - x^2)$.

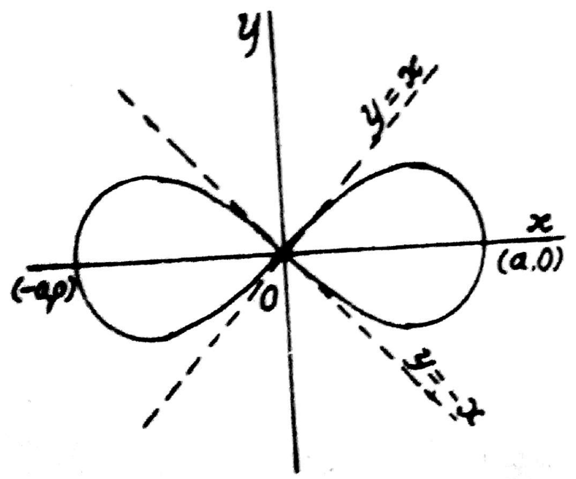


Fig. 110

(i) Curve is symmetrical about both the axes.

(ii) It passes through the origin and $y = \pm x$ are the tangents at the origin.

(iii) Curve meets the x axis at $(a, 0)$ and $(-a, 0)$ and since at $x = \pm a$, $\frac{dy}{dx}$ is infinite, the tangents at these points are parallel to y axis.

(iv) For values of $x > a$ and $x < -a$, y^2 being negative the curve entirely lies between $x = -a$ and $x = a$. The curve has the form as shown in Fig. 110.

(5) *Trace the curve* $y^2 = (x - \alpha)(x - \beta)(x - \gamma)$, *for different values of* α, β, γ *which are assumed as positive.*

We will consider here only three important cases viz.

(I) $\alpha < \beta < \gamma$ (II) $\beta = \gamma$ and $\alpha < \beta$ (i. e. γ) (III) $\alpha = \beta = \gamma$.

Case I $\alpha < \beta < \gamma$:–

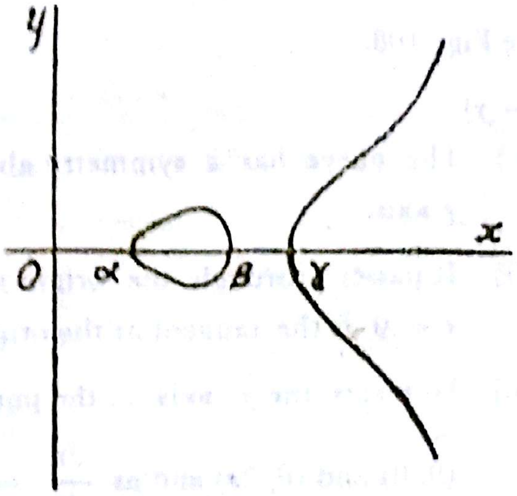


Fig. 111

(i) Curve is symmetrical about x axis.

(ii) It meets the x axis at points $(\alpha, 0)$ $(\beta, 0)$ and $(\gamma, 0)$

(iii) For values of $x < \alpha$ and $\beta < x < \gamma$. y^2 being negative, curve does not lie in the regions for which $x < \alpha$ and $\beta < x < \gamma$.

(iv) $\dfrac{dy}{dx}$ is infinite at $x = \alpha, \beta, \gamma$ i.e. the tangents at these points are parallel to y-axis.

(v) For values of $x > \gamma$, y increases as x increases. Fig. 111 gives the form of the curve.

Case II $\beta = \gamma$ and $\alpha < \beta \ (= \gamma)$:–

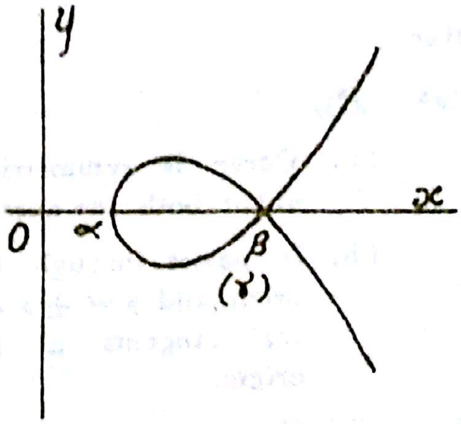


Fig. 112

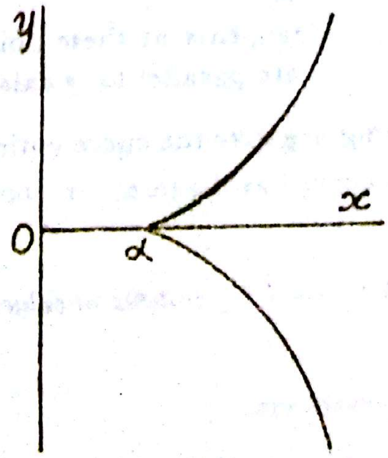
The curve is given by

$$y^2 = (x - \alpha)(x - \beta)^2.$$

With the similar analysis as above, the curve is as sketched in the Fig. 112

Case III $\alpha = \beta = \gamma$:– The equation of the curve is

$$y^2 = (x - \alpha)^3.$$



The sketch of the curve is as shown in the adjoining figure.

Fig. 113

Forms of some of the curves are given here for the reference, however, the discussion is left to the students.

(1) $ay^2 = x(a^2 - x^2)$

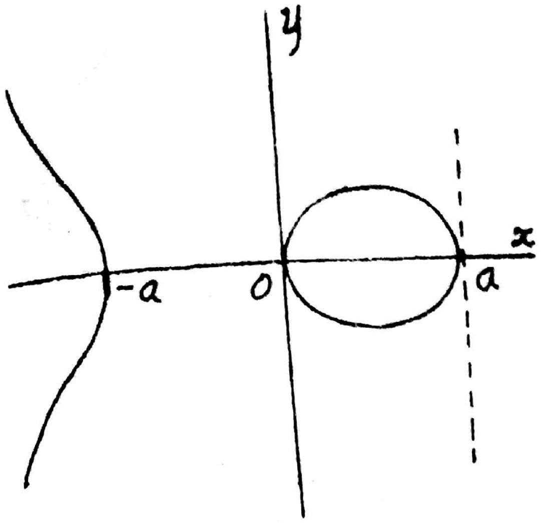


Fig. 114

(2) $ay^2 = x(a^2 + x^2)$

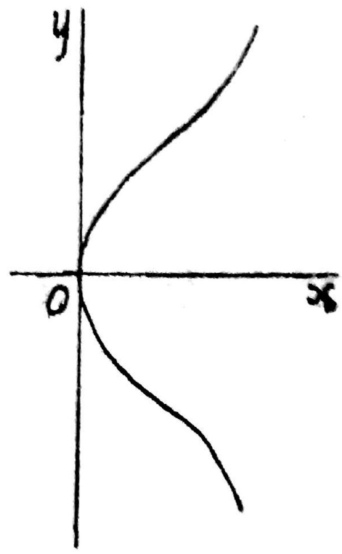


Fig. 115

(3) $3ay^2 = x(x - a)^2$

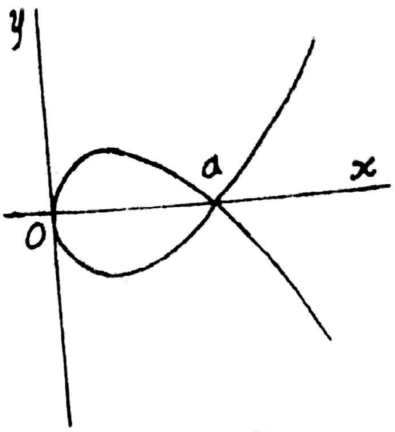


Fig. 116

(4) $3ay^2 = x^2(a - x)$

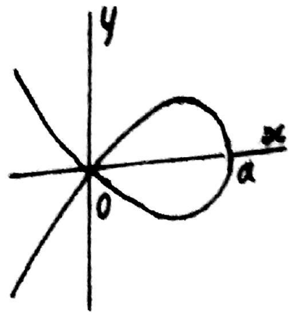


Fig. 117

(5) $27\,ay^2 = 4(x - 2a)^3$
[evolute of parabola]

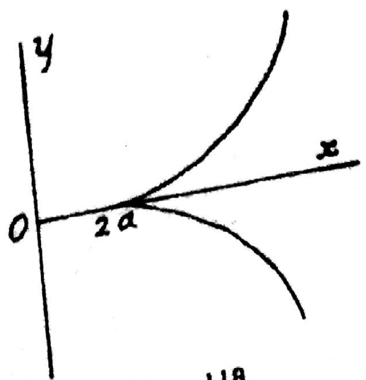


Fig 118

(6) $a^2y^2 = x^2(a^2 - x^2)$

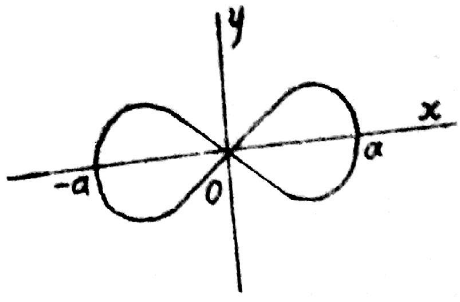


Fig. 119

(7) $a^2y^2 = x^2(2a - x)(x - a)$

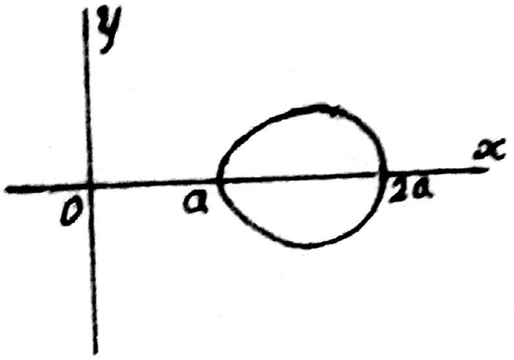


Fig. 120

(8) $y^2x = a^2(a - y)$

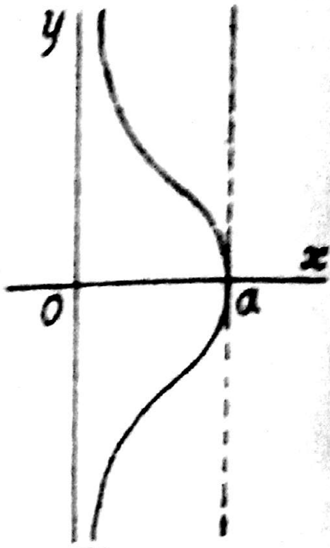


Fig. 121

(9) $y^2 = x^5(2a - x)$

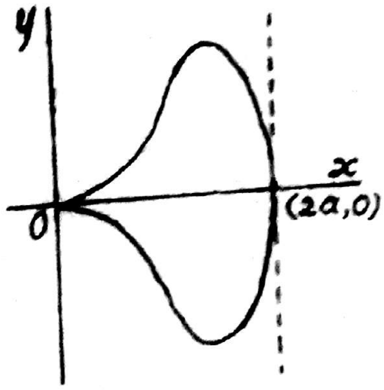


Fig. 122

(10) $x^2y^2 = a^2(y^2 - x^2)$

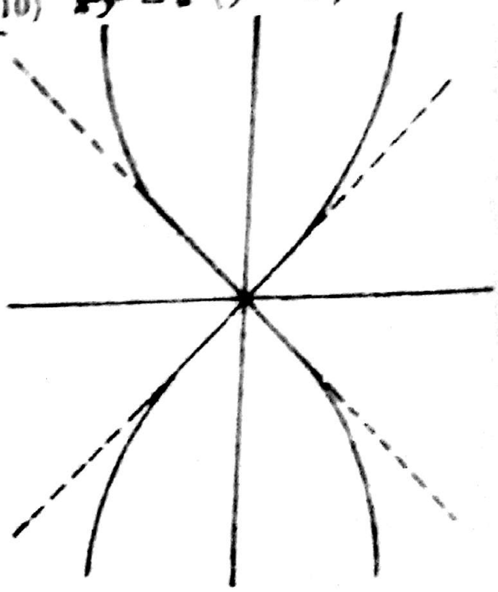


Fig. 123

(11) $y^2(a^2 - x^2) = a^2x$

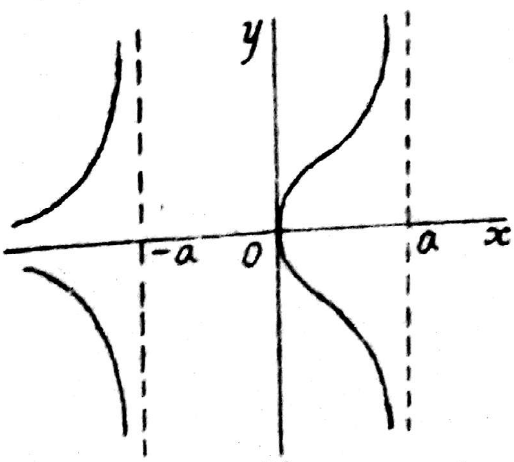


Fig. 124

(12) $xy^2 + (x + a)^2(x + 2a) = 0$

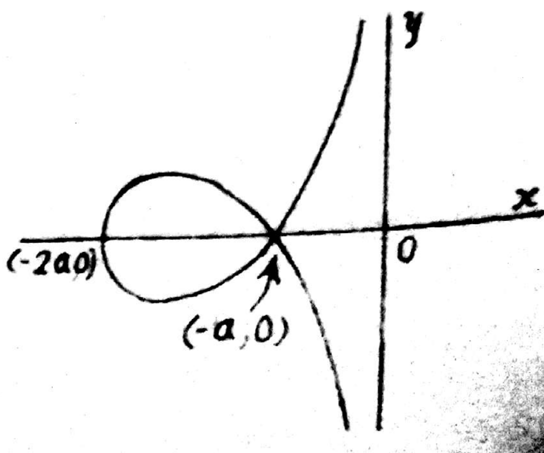


Fig. 125

(13) $y(x^2 + 4a^2) = 8a^3$

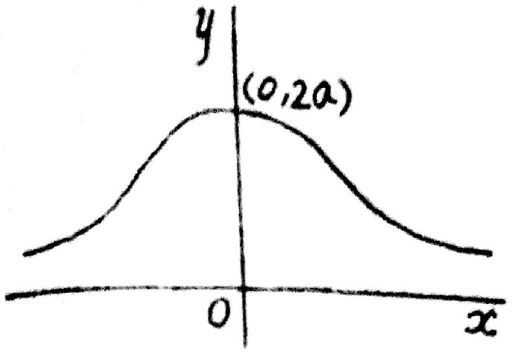


Fig. 126

(14) $x(x^2 + y^2) = a(x^2 - y^2)$

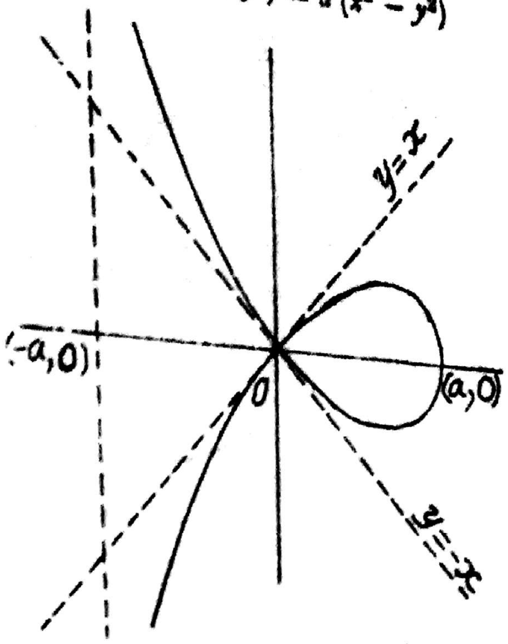


Fig. 127

(15) $x^2(x^2 + y^2) = a^2(x^2 - y^2)$

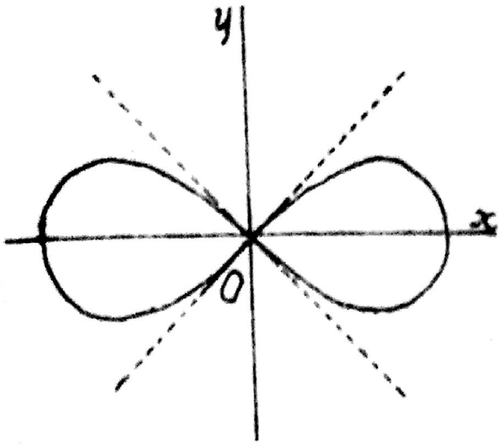


Fig. 128

(16) $x^4 + y^4 = 4axy^2$

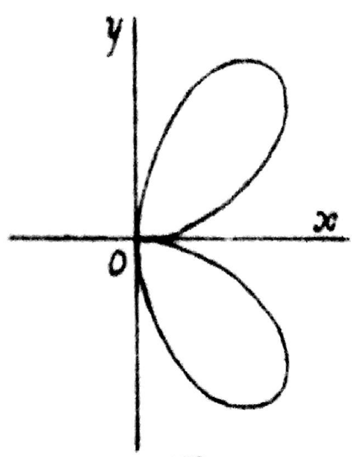


Fig. 129

(17) $x^4 + y^4 = 2a^2xy$

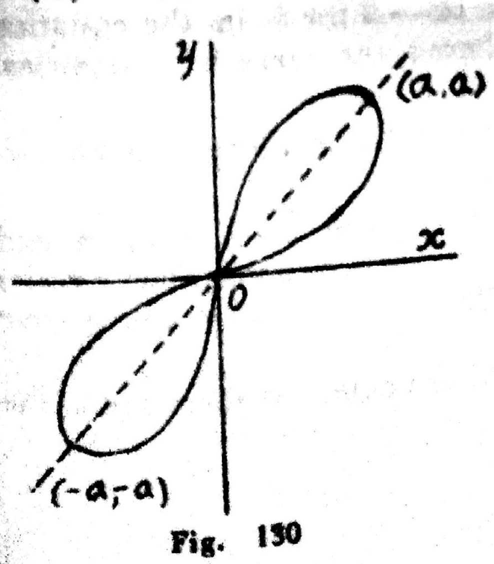


Fig. 130

(18) $x^5 + y^5 = 5ax^2y^2$

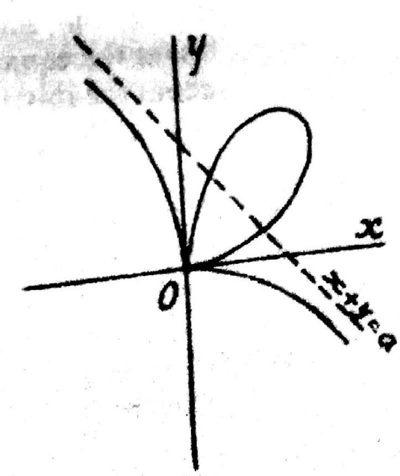


Fig. 131

(19) $x^5 + y^5 = 5a^2x^2y$

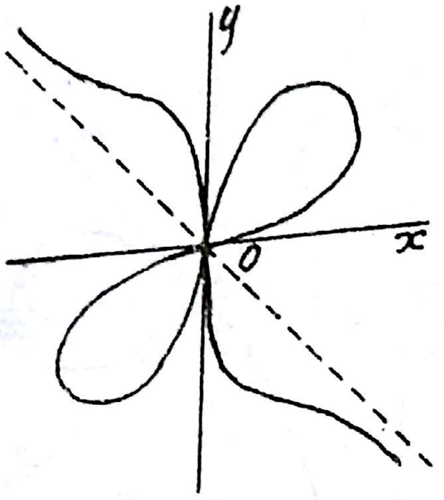


Fig. 132

(20) $x^6 + y^6 = a^2x^2y^2$

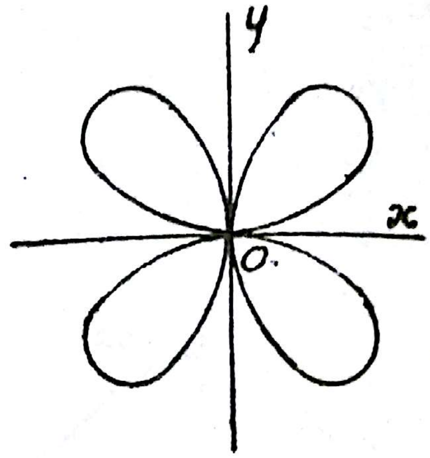


Fig. 133

(21) $y = \dfrac{x}{1 + x^2}$

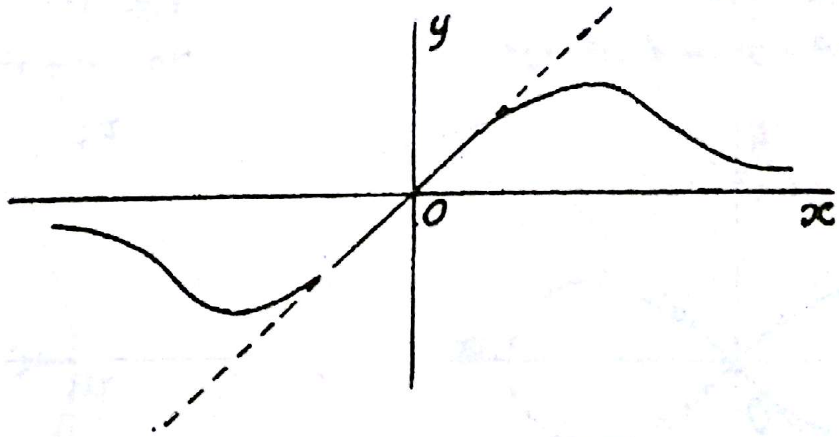


Fig. 134

13·2. Polar equation :—

The hints given below should enable the students to sketch a curve given by polar equation in simple cases.

(A) Find the symmetry if any

(i) If the substitution of $-\theta$ for θ in the equation leaves the equation unaltered, the curve is symmetrical about the initial line.

(ii) If the powers of r are even the curve is symmetrical about the pole (origin).

(B) Form the table of values of r for both positive and negative values of θ and thence note how r varies with θ. Find in particular the values of θ which give $r = 0$ and $r = \infty$

(C) Find $\tan \phi$. This will indicate the direction of the tangent.

(D) Sometimes from the nature of the equations it is possible to ascertain the value of r or θ, that are confined between certain limits. For example for the curve $r = a \sin n\theta$, r must lie between the limits 0 and a, and the curve must lie within the circle of radius a.

(E) Transform into cartesian if necessary and adopt the methods given before.

(I) Type $\mathbf{r = a \sin n\theta}$

(1) Trace the curve $r = a \sin 5\theta$

(i) Table of values :—

θ	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$
r	0	a	0	$-a$	0	a	0	$-a$	0

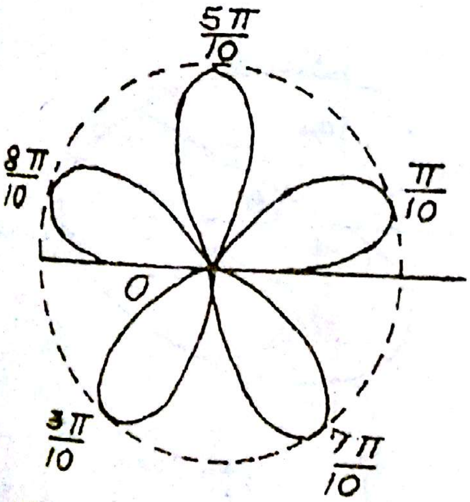


Fig. 135

Thus we see that r is never greater than a and the curve lies wholly within a circle of radius a.

(ii) $\tan \phi = \dfrac{1}{5} \tan 5\theta$, which is zero when $r = 0$ and is infinite when $r = \pm a$ i. e. the tangents at the points where $r = 0$, are coincident with radius vectors and at points $r = \pm a$, the tangents are perpendicular to radius vectors.

Hence the curve consists of a series of similar loops as shown in the figure 135.

Note :— By taking particular cases for different values of n for the curve $r = a \sin n\,\theta$, we observe that

(i) there are n loops, if n is odd.

(ii) there are $2n$ loops, if n is even.

Type $\mathbf{r = a \cos n\,\theta}$:—

This family of curves are just similar to the previous type and the curve has n loops if n is odd and has $2n$ loops if n is even.

(II) Type $\mathbf{r^n = a^n \cos n\,\theta}$:—

The family of the curves given by above type for different values of n, is of importance as it contains six well known curves for values of $n = \pm 1, \pm 2, \pm \frac{1}{2}$. These curves are sketched here but it is left to the students for detailed discussion.

(ii) *Archimedes spiral* $r = a\theta$:—

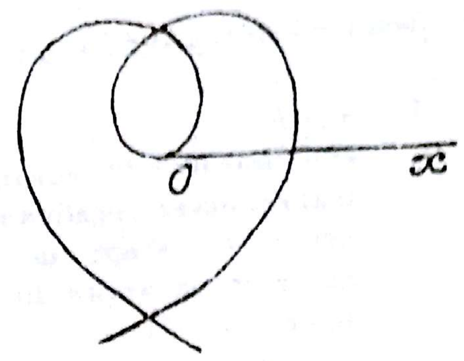


Fig. 144

The shape of the curve is as shown in the adjoining figure.

The main property of this spiral is that if a circle of radius a be drawn with the centre at the pole, the radius vector of the curve is equal to the arc of this circle measured from the initial line to the point in which the radius vector cuts the circle.

(iii) *The Reciprocal spiral* $r\theta = a$

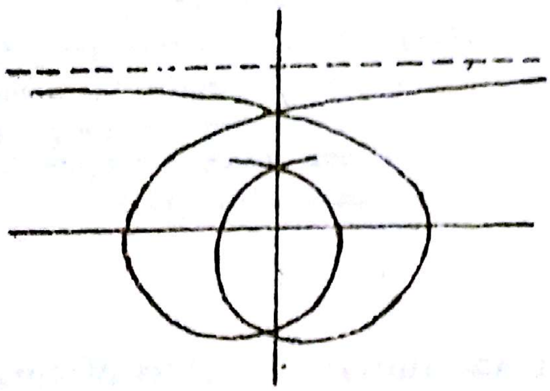


Fig. 145

The property of the curve is that for a circle of any radius with centre at the pole, the arc intercepted on it by the points where it is cut by the initial line and the curve is of constant length.

(iv) *Lituus or socoll* $r\sqrt{\theta} = a$

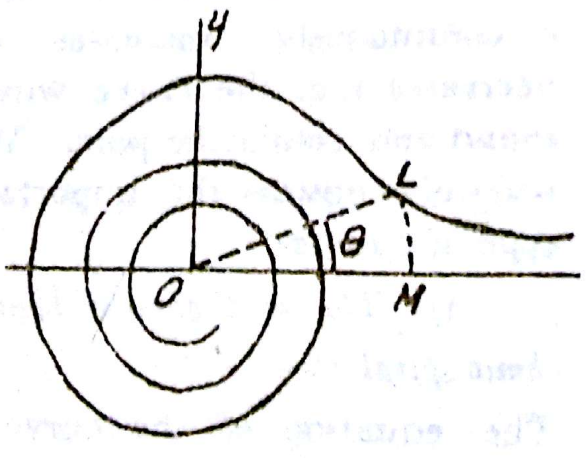


Fig. 146

The curve is as shown in the fig. 146. The property is that the area of circular sector OLM bounded by the initial line and the radius vector OL with the radius of circle as OL is constant. For,

$$\text{Area} = \tfrac{1}{2} r^2\theta = \tfrac{1}{2} \cdot \frac{a^2}{\theta}\,\theta \qquad [\text{as } r\sqrt{\theta} = a]$$

$$= \tfrac{1}{2} a^2 = \text{constant.}$$

It is found convenient to make use of transformation to polar in case of cartesian curves whose equations are not solvable for x and y, but whose transformed polar equations are solvable for r. This is illustrated in the following problem.

Example. Trace the curve $x^6 + y^6 - 5a^2x^3y = 0$

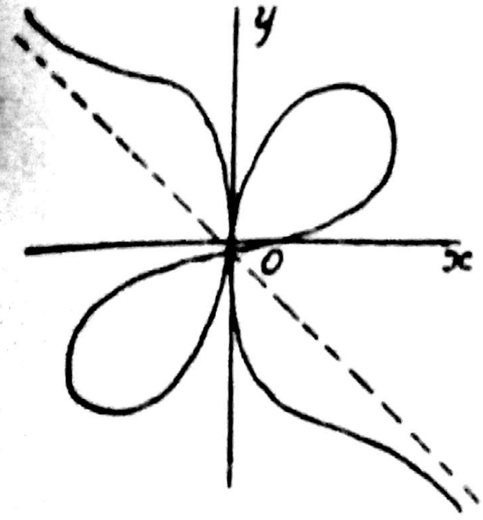


Fig. 147

Substituting $x = r\cos\theta$, $y = r\sin\theta$, the polar equivalent of the given equation is

$$r^2 = 5a^2 \frac{\sin\theta\cos^3\theta}{\cos^6\theta + \sin^6\theta}$$

The table of values is given by :—

$\theta =$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$r =$	0	$\sqrt{5}\,a$	0	∞	0	$\sqrt{5}\,a$	0	∞

With the help of the above table and methods adopted in previous problems, the curve represented by the given equation is shown in the fig. 147.

STANDARD CURVES

13·4. The common catenary :—

A curve in which a perfectly flexible and uniform heavy string hangs under gravity is called *Catenary*.

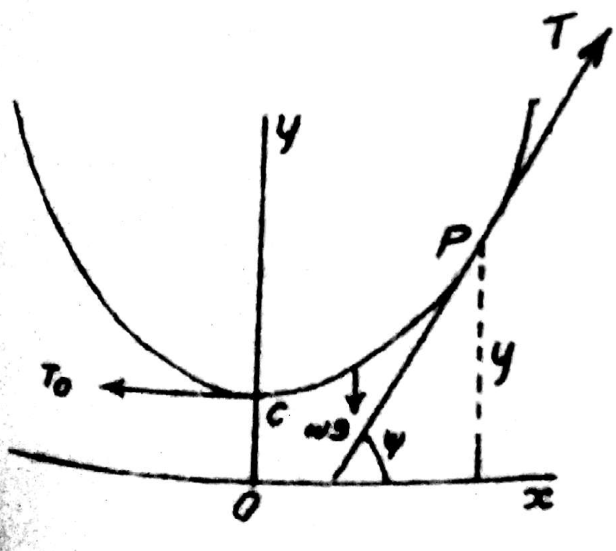


Fig. 148

Let C be the lowest point of the string of uniform weight w per unit length and T and T_0 be the tensions at P and C respectively.

For equilibrium of the portion CP under the action of gravity, we have [ref. fig. 148].

$T\cos\psi = T_0$ and

$T\sin\psi = ws$

$$\therefore \quad \tan\psi = \frac{ws}{T_0} = \frac{s}{c} \quad [\text{where } T_0 = \text{constant} = wc]$$

$$\therefore \quad \frac{dy}{dx} = \tan\psi = \frac{s}{c}$$

Differentiating w. r. t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{c} \cdot \frac{ds}{dx} = \frac{1}{c}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Adjusting and integrating, we get

$$\int \frac{\dfrac{d^2y}{dx^2}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}\, dx = \frac{1}{c}\, x + k, \ (k = \text{const.})$$

i. e. $\sinh^{-1}\left(\dfrac{dy}{dx}\right) = \dfrac{x}{c}$ [as $k = 0$, for $\dfrac{dy}{dx} = 0$ when $x = 0$]

$$\therefore \quad \frac{dy}{dx} = \sinh\frac{x}{c}$$

Integrating, we have $y = c\cosh\dfrac{x}{c} + k'$

Choosing the origin at a depth c below the point C, we have $y = c$ when $x = 0$ and hence $k' = 0$.

Thus the equation of the curve with horizontal line Ox (*directrix*) at a depth c (*parameter*) below the lowest point C (*vertex*) as x-axis and the line through the vertex C perpendicular to Ox as y-axis is given by

$$y = c\cosh\frac{x}{c} \quad \ldots \quad \ldots \quad (1)$$

Properties :—

(i) $s = c\tan\psi$ (Intrinsic equation) (ii) $y^2 = c^2 + s^2$

13.5. The Cissoid :—

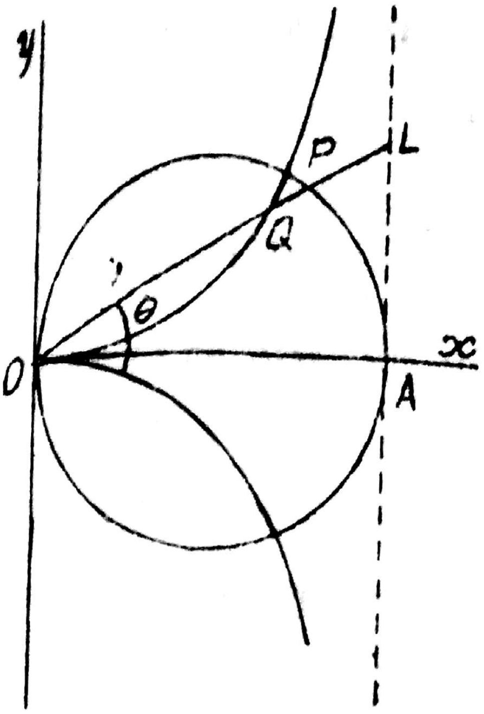


Fig. 149

Let OPA be a circle with OA as a diameter and let a st. line through O intersect the circle in P and the tangent at A in L. Then the locus of a point Q such that

$$OQ = PL$$

is known as *Cissoid.*

Equation :—

Taking OA as the initial line and $L\hat{O}A = \theta$, we have

$$OP = 2a \cos\theta$$

[$OA = 2a$ = diameter of the circle]

and $OL = 2a \sec\theta$

Hence $PL = OL - OP = 2 [\sec\theta - \cos\theta]$

$$= \frac{2a \sin^2\theta}{\cos\theta} = OQ \ (= r), \text{ by definition of locus.}$$

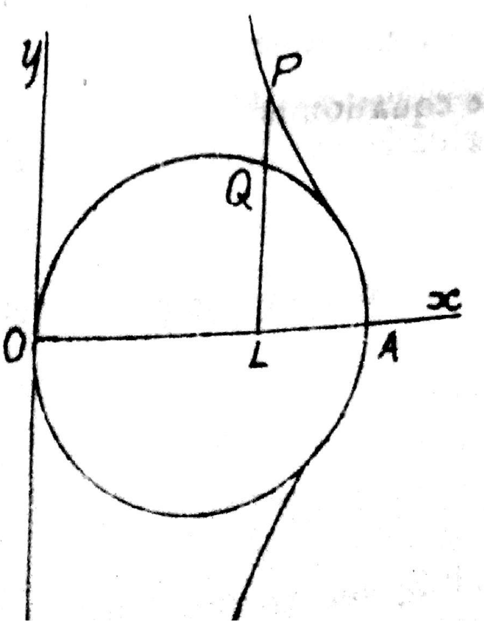
∴ Polar equation of the *Cissiod* is

$$r \cos\theta = 2a \sin^2\theta \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Transforming into cartesian form, we get

$$(2a - x)\, y^2 = x^3 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

13.6. Witch of Agnesi :—



Let OQA be a circle of OA ($= 2a$) as diameter. If P is a point on any line LP drawn perpendicular to OA and intersecting the circle in Q, such that

$$\frac{LQ}{LP} = \frac{OL}{OA} \quad \ldots \quad \ldots \quad (i)$$

then the locus of P is known as *Witch.*

Equation :—

Let the co-ordinates of P, Q ... respectively

with reference OA and a line through O perpendicular to OA as co-ordinate axes i. e.

$$LQ = y_1, \; LP = y, \; OL = x, \; OA = 2a$$

Hence as P lies on the locus we have

$$\frac{LQ}{LP} = \frac{OL}{OA} \text{ i. e. } \frac{y_1}{y} = \frac{x}{2a} \quad \ldots \quad \ldots \quad \text{(ii)}$$

But $y_1^2 = LQ^2 = OL.LA = x\,(2a - x)$ ... ... (iii)

Substituting y_1 from (ii) in (iii), the cartesian equation of the *Witch* is given by

$$y^2 = \frac{4a^2\,(2a - x)}{x} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

The shape of the curve is given in the Fig 150.

13·7. The Folium of Descartes :—

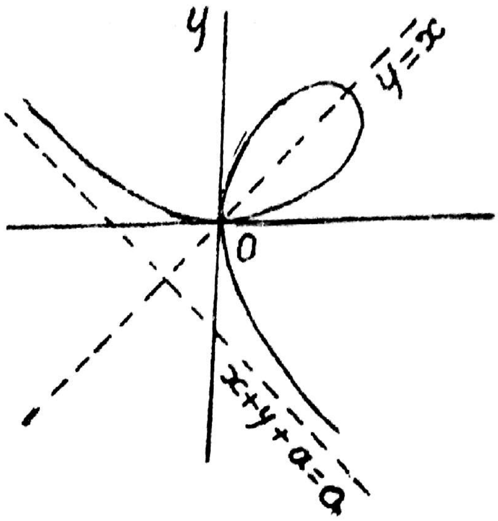


Fig. 151

The cartsian equation of the curve is

$$x^3 + y^3 = 3axy \quad \ldots \quad (5)$$

(i) Curve is symmetrical about $y = x$.

(ii) Passes through the origin and touches the axes at the origin.

(iii) There is a loop in the first quadrant and $x + y + a = 0$ is the asymptote.

(iv) The parametric form of the equation is

$$x = \frac{3at}{1 + t^3}, \; y = \frac{3at^2}{1 + t^3}$$

(v) The curve can easily be traced from its polar equation, viz.

$$r = \frac{3a \sin\theta \cos\theta}{\cos^3\theta + \sin^3\theta}$$

13·8 Roulettes :—

If one curve rolls without sliding on another fixed curve the locus of fixed point on the rolling curve is known as *roulette.*

We will consider briefly some curves of this type which have practical importance, viz.

(i) Cycloid (ii) Epi-and hypocycloids.

(1) Cycloid : –

When a circle rolls in a plane along a given straight line the locus traced out by any fixed point on the circumference of the rolling circle is called a *cycloid*.

Equation : –Let the circle P′TPG with centre C and radius a roll on a line DGA which is known as the base of the cycloid and A be the position of fixed point P when the circle touches the line AGD at the fixed point and O be the position of P when the diameter PP′ through P is vertical.

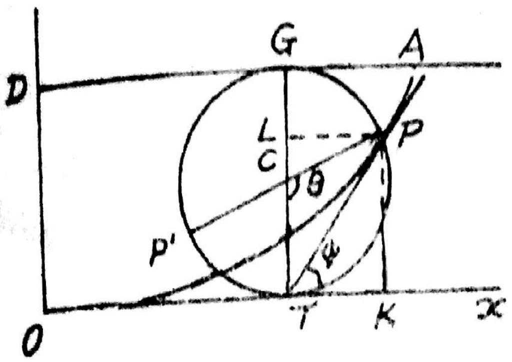


Fig. 152

As the circle rolls on along AGD, PT is the direction along which P moves i. e. PT represents the tangent to the path of P at P.

We have arc PG = AG
and arc P′G = GD [P′ being the extremity of the diameter PP′].

Let a line OK through O parallel to the line DGA be x-axis and line OD perpendicular to OK be y-axis.

and PĈT = θ

Let the co-ordinates of P be (x,y)

Then we have

$$x = OT + TK = OT + PL$$
$$= DG + CP \sin(\pi - \theta)$$
$$= a\theta + a \sin\theta \; [\text{For arc } GD = \text{arc } P'G = a\theta]$$
$$= a(\theta + \sin\theta)$$

$$\text{and } y = PK = CT + CL$$
$$= a + CP \cos(\pi - \theta)$$
$$= a(1 - \cos\theta)$$

Thus the parametric equation of the cycloid is

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta) \quad \dots \quad \dots \quad \dots \quad (3)$$

Note : —Depending upon the choice of axes, we have different forms of equation to the cycloid and as students should not get confused with the form of the curve for a particular equation of the cycloid, the three types of equations with the forms of the curves they represent are given here.

(i)

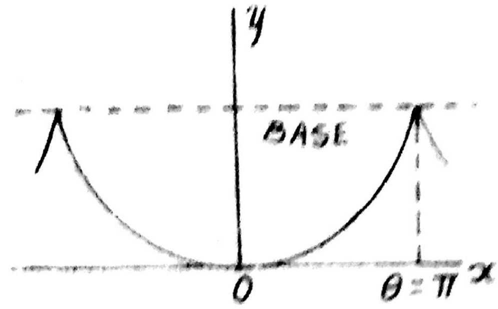


Fig. 153

$$x = a(\theta + \sin\theta)$$
$$y = a(1 - \cos\theta)$$

(ii)

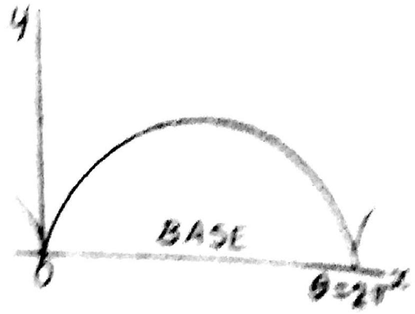


Fig. 154

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

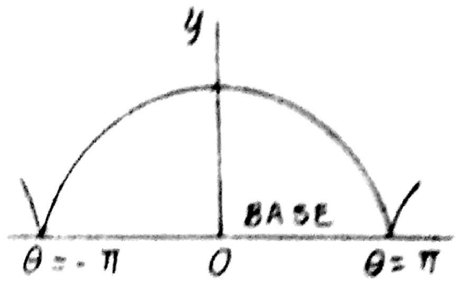


Fig. 155

$$x = a(\theta + \sin\theta)$$
$$y = a(1 + \cos\theta)$$

Properties of the curve : —

(i) $s = 4a \sin\psi$ (*intrinsic equation*).

(ii) Area of the cycloid between the base and the curve $= 3\pi a^2$.

(iii) Volume of revolution about the base $= 5\pi^2 a^3$.

(iv) Height of C. G. above the base $= \dfrac{5a}{6}$.

If the fixed point is not on the circumference but at a distance b from the centre, the curve traced by it is known as trochoid whose equations are

$$x = a\theta + b\sin\theta,\ y = a - b\cos\theta.$$

(2) The epi-and hypo-cycloids :—

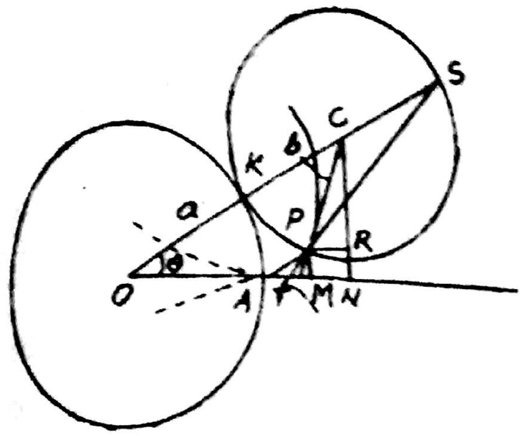


Fig. 156

The locus of a fixed point on the circumference of a circle rolling on the circumference of another fixed circle is called an *epicycloid* for rolling on the outside of the fixed circle, and a *hypocycloid* for rolling on the inside.

Equations :—

(*a*) *Epicycloid* :—Let P be the position of the fixed point after the circle C has rolled from A to K so that A is the starting point.

Hence arc KA = arc KP

If $K\hat{O}A = \theta$, $OK = a$, $KC = b$, then

$$K\hat{C}P = \frac{\text{arc KP}}{b} = \frac{\text{arc KA}}{b} = \frac{a\theta}{b} \qquad \dots \qquad \dots \qquad \text{(I)}$$

Similarly $C\hat{P}R = C\hat{L}N = K\hat{C}L + K\hat{O}A$
[Ext. angle of Δ COL]

$$= \theta + \frac{a\theta}{b} = \left(\frac{a+b}{b}\right)\theta.$$

$$\therefore\quad x = OM = ON - PR = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta$$

$$y = PM = CN - CR = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta.$$

Thus the equations to the epicycloid are

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta \\ y &= (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta \end{aligned}\right\} \quad \ldots \quad (6)$$

Since K is momentarily at rest and $K\hat{P}S = 90°$, SP is the tangent at P and

$$\psi = S\hat{T}N = \theta + P\hat{S}K = \theta + \frac{a}{2b}\theta = \left(\frac{a+2b}{2b}\right)\theta$$

Cor :— If $a = b$, we get the cardioid, whose equations are

$$x = 2a\cos\theta - a\cos 2\theta,\ y = 2a\sin\theta - a\sin 2\theta \quad \ldots \quad (7)$$

(*b*) *Hypocycloid* :— By changing b to $-b$ or by the similar method, the equations to *hypocycloid* are

$$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\right)\theta$$

$$y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\right)\theta.$$

If $b = \frac{1}{4}a$, the curve is four cusped hypocycloid whose equations are given by

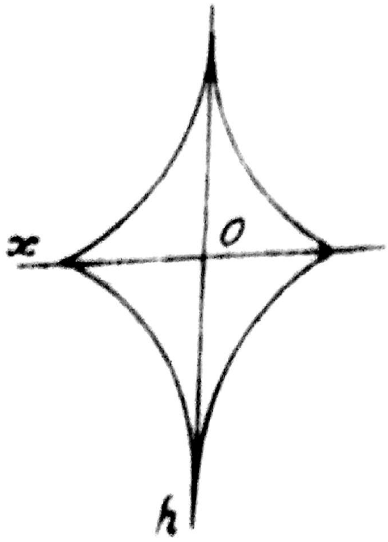


Fig. 157

$$\left.\begin{aligned} x &= \frac{3}{4}a\cos\phi + \frac{a}{4}\cos 3\phi = a\cos^3\phi \\ y &= \frac{3}{4}a\sin\phi - \frac{a}{4}\sin 3\phi = a\sin^3\phi \end{aligned}\right\} \quad \ldots \quad (8)$$

The Fig. 157 represents the curve given by the equations (8).

CHAPTER 16

MULTIPLE INTEGRALS

16.1. Double Integral : Introduction and Notation :—

It is presumed that the students are familiar with "the limit of a sum as an integral."

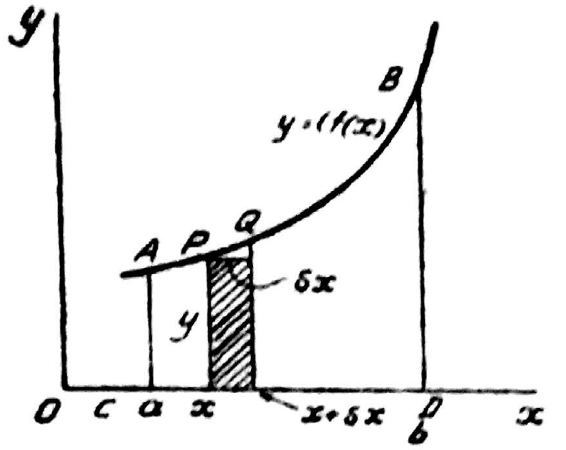


Fig. 170

Thus if P (x, y), Q $(x+\delta x, y+\delta y)$ are two adjacent points on the curve $y=f(x)$, the area of the elementary strip formed by the ordinates at P and Q, the curve and the x-axis is, to the first order of smallness, given by $y\delta x$. Forming such expressions for the elementary areas between $x=a$, and $x=b$, taking the sum and proceeding to the limit as $\delta x \to 0$, we have the area ABCD given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x$$ and this is expressed as

$$\int_a^b y\,dx$$

Thus $$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x = \int_a^b y\,dx.$$

Let us now consider the integration of a function of two variables over a given area. To make the idea clear, we shall consider a plane lamina in the xOy plane, the surface density σ of which is a function of the position of the point P (x, y). Thus the surface density $\sigma = f(x, y)$.

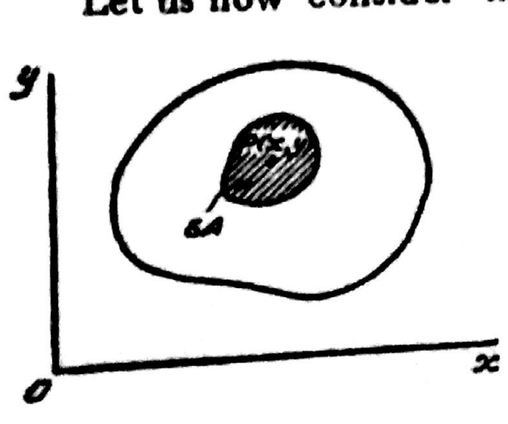


Fig. 171

To find the mass of the lamina, we shall take a small area δA about the point P (x, y).

The mass of this elementary area is $f(x, y)\,\delta A$. To find the total mass of the lamina, we shall find out expressions such as $f(x, y)\,\delta A$, all over the lamina, form the sum $\Sigma f(x, y)\,\delta A$, and to be more accurate, δA must be taken as small as possible. That is

The mass of the lamina $= \lim\limits_{\delta A \to 0} \Sigma f(x, y)\,\delta A$ ... ... (1)

where the summation extends all over the lamina.

Let us take δA in a more convenient way so that the summation in (1) can be carried out.

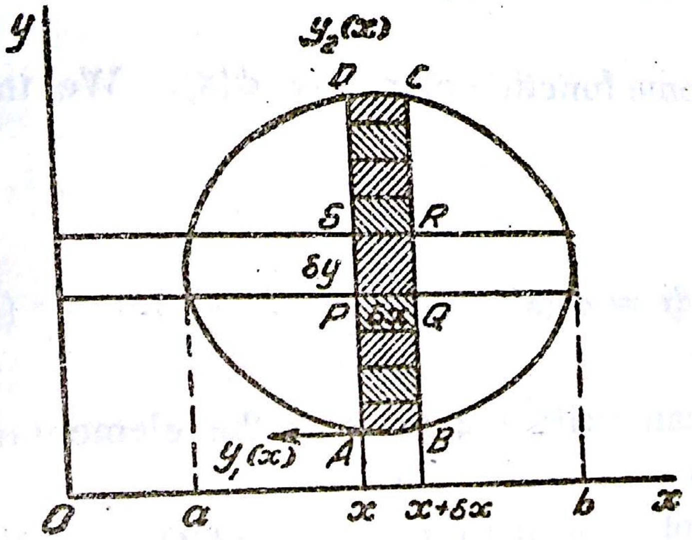


Fig. 172

Divide the lamina by a system of straight lines parallel to the x and y axis into a mesh of elementary rectangles. Take the rectangle with one corner at P (x, y).

Then the area of rectangle PQRS $\delta A = \delta x . \delta y$

and the mass of the elementary rectangle $= f(x, y)\,\delta x \delta y$.

By (1), the mass of the lamina M is

$$M = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \Sigma\Sigma f(x, y)\,\delta x \delta y \quad \text{.. ... ..} \quad (2)$$

We shall evaluate the expression on the r. h. s. of (2) in a systematic way.

Taking the sum of $f(x, y)\,\delta x\,\delta y$ over the strip ABCD, we have for the mass of the elementary strip ABCD

$$= \lim_{\delta y \to 0} \sum_{A}^{D} f(x, y)\,\delta x\,\delta y \quad \text{... .. ..} \quad (3)$$

where in this summation we note that x and δx are constants. We can therefore write (3) as

$$= \delta x \lim_{\delta y \to 0} \sum_{y_A}^{y_D} f(x, y)\,\delta y \quad \text{.. ... ...} \quad (4)$$

and by the introductory remarks on the limit of the sum as an integral, we write (4) as

$$= \delta x \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy \quad .. \quad ... \quad ... \quad (5)$$

where $y_1(x)$, and $y_2(x)$ are the values of y at A and D and both depend on the position of the ordinate, that is on x.

It is to be remembered in the integral of (5) that x is to be regarded as a constant in the integration w. r. t. y, and since the limits of the integral are functions of x,

so $\int_{y_1(x)}^{y_2(x)} f(x, y)\, dy$ will be some function of x, say $\phi(x)$. We thus say that let

$$\int_{y_1(x)}^{y_2(x)} f(x, y)\, dy = \phi(x) \quad ... \quad ... \quad ... \quad (6)$$

so that from (5), we can write the mass of the elementary strip ABCD as $[\phi(x).\, \delta x]$

Next taking the mass of each strip such as ABCD, parallel to the y-axis, over the area of the lamina, we have

$$\text{Mass of the lamina} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \phi(x)\, \delta x.$$

$$= \int_{x=a}^{x=b} \phi(x)\, dx. \quad ... \quad ... \quad (7)$$

Substituting for $\phi(x)$ from (6) in (7), we get

$$\text{Mass of the lamina} = \int_{x=a}^{x=b} \left\{ \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy \right\} dx \quad ... \quad (8)$$

The expression on the r. h. s. of equation (8) is called a double integral for obvious reason and is written in various ways as follows

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy\, dx. \quad ... \quad ... \quad (9a)$$

or
$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\, dx\, dy \qquad \ldots \qquad \ldots \qquad (9b)$$

where the integral signs are written in order of integration taken from the right,

or
$$\int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy \qquad \ldots \qquad \ldots \qquad (9c)$$

This last way of writing the integral is more convenient, as it expresses clearly the order in which the integration is performed i. e. we first integrate w. r. t. y considering x as a constant and then we integrate w. r. t. x. It may also be noted that when *we take the elementary strips parallel to the y-axis, we first integrate w. r. t. y.*

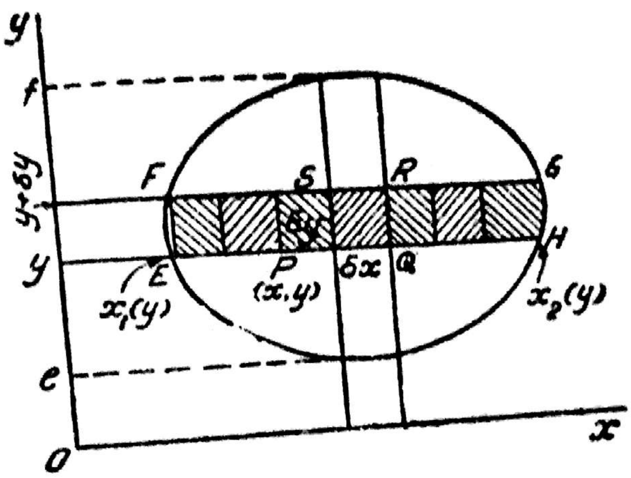


Fig. 173

If instead of taking the elementary strip parallel to the y-axis we take it parallel to the x-axis such as EFGH shown in the adjacent figure, we have by a similar reasoning to the above

$$\text{Mass of the lamina} = \int_e^f dy \int_{x_1(y)}^{x_2(y)} f(x, y)\, dx \qquad \ldots \qquad \ldots \qquad (10)$$

in which we have to first integrate w. r. t. x and then w. r. t. y, thus changing the order of integration. Both the integrals (9) and (10) represent the mass of the lamina and so are equal. The total area of the lamina is known as the region of integration.

The function $f(x, y)$ was considered as the surface density of the lamina, just for the sake of understanding clearly the idea of double integral. However $f(x, y)$ may be any function of the position of a point in the loop-area, and the double integral of this function over the area of the loop is given by (9) or (10) that is

$$\boxed{\int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy \quad \text{or} \quad \int_e^f dy \int_{x_1(y)}^{x_2(y)} f(x, y)\, dx} \qquad \ldots \qquad \ldots \qquad (11)$$

16·2. Evaluation of Double integrals; Change of the order of integration :–

The method of evaluating the double integrals (11) is actually clear from the theory developed in the previous article. We note that in the evaluation of the double integrals, we integrate first w. r. t. one variable (y or x depending upon the limits and the elementary strip) and co sidering the other variable as constant and then integrate with respect to the remaining variable.

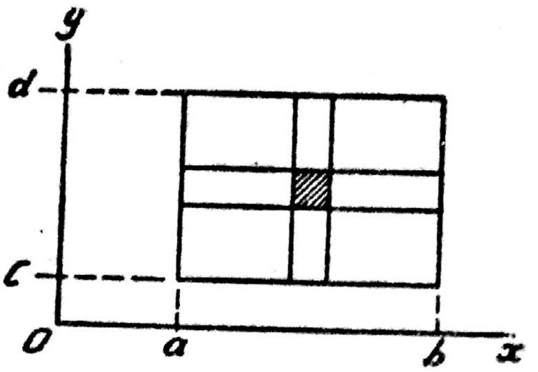


Fig. 174

If the limits of integrations are constants such as in the region of integration being a rectangle, then the change in the order of integration does not require the change of the limits of integration. Thus from the adjacent figure, we see that

$$\int_a^b dx \int_c^d f(x,y)\, dy = \int_c^d dy \int_a^b f(x,y)\, dx \quad \ldots \quad \ldots \quad (12)$$

But if the limits be variable, as in the general case taken in art. 16·1 then in changing the order of integration a corresponding change is to be made in the limits of integration as seen from (11). Sometimes, in changing the order of integration we are required to split up the region of integration and the new integral is expressed as a sum of a number of double integrals. The examples solved below will make these ideas clear. The change of the order of integration is sometimes convenient in the evaluation of the double integrals. This is also illustrated in some of the problems solved below. In changing the order of integration, it is convenient to draw rough sketch of the region of integration, which will help to fix up the new limits of integration.

Example 1. Evaluate $\int (x^2 - y^2)\, dA$ over the area of the triangle whose vertices are at the points (0, 1), (1, 1) and (1, 2).

The equations of the sides of the triangle whose vertices are at A (0, 1), B (1, 1), C (1, 2) are

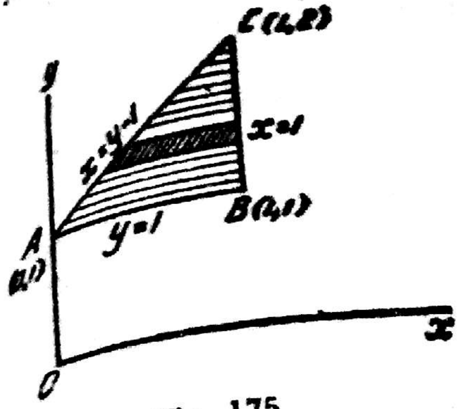


Fig. 175

$x = 1, y = 1$, and $x = y - 1$ .. .. (i)
as shown in the figure.

If we take an elementary strip parallel to the x-axis, we will be integrating the given function with respect to x. The ends of this strip are bounded by the lines $x = y - 1$ and $x = 1$, so that these are the limits of integration with respect to x. Next we integrate w.r.t. y from $y = 1$ to $y = 2$, which then covers the whole area of the triangle ABC.

Thus if

$$I = \int (x^2 - y^2)\, dA \text{ taken over the area of the triangle ABC}$$

Then,

$$I = \int_1^2 dy \int_{y-1}^{1} (x^2 - y^2)\, dx \quad .. \quad .. \quad .. \quad \text{(ii)}$$

To evaluate the first integral, we regard y as a constant,

$$\therefore \quad I = \int_1^2 dy \left[\frac{x^3}{3} - y^2 x\right]_{y-1}^{1}$$

$$= \int_1^2 \left\{\frac{1}{3} - y^2 - \frac{(y-1)^3}{3} + y^2(y-1)\right\} dy$$

$$= \int_1^2 \left\{\frac{1}{3} - 2y^2 - \frac{(y-1)^3}{3} + y^3\right\} dy$$

$$= \left[\frac{y}{3} - \frac{2y^3}{3} - \frac{(y-1)^4}{12} + \frac{y^4}{4}\right]_1^2$$

$$= \left[\frac{2}{3} - \frac{16}{3} - \frac{1}{12} + 4 - \frac{1}{3} + \frac{2}{3} - \frac{1}{4}\right]$$

$$= -\frac{2}{3}$$

It will be interesting to try the above example by taking elementary strips parallel to the y-axis, which is left to the students as an exercise leading to the same result as above.

Example 2. Evaluate $\displaystyle\int_0^a dy \int_0^{a - \sqrt{a^2 - y^2}} \frac{xy \log(x + a)}{(x - a)^2}\, dx.$

In the integral as it stands, the integration is first w. r. t. x, and this integration, as is clear, is complicated. As integration w. r. t. y is simple, we therefore change the order of integration, for which sake we find out the region of integration for the given problem.

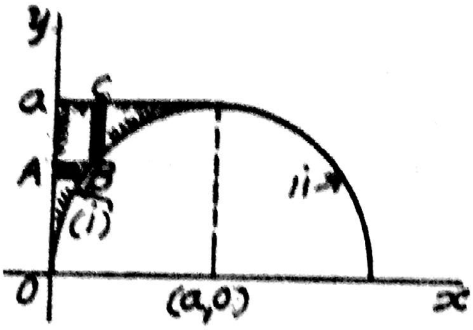


Fig. 176

In the given integral, where the integration is first w. r. t. x, the elementary strips are parallel to the x-axis and these strips extend from $x=0$ (i. e. the y-axis) to $x=a-\sqrt{a^2-y^2}$ i. e. to the boundary of the circle $(x-a)^2+y^2=a^2$. Moreover as $x=a$ *minus* $\sqrt{a^2-y^2}$, it extends upto the side (i) of the circle and not upto (ii) for which $x=a$ *plus* $\sqrt{a^2-y^2}$. An elementary strip such as this, is shown in the figure by AB. Next we integrate w. r. t. y from $y=0$ to $y=a$ and so the strips such as AB, bounded on one side by the y-axis and on the other by the circumference of the circle are taken from $y=0$ to $y=a$. Thus the region of integration is the shaded part in the figure.

If we change the order of integration, integrating first w. r. t. y, then the elementary strip is parallel to the y-axis, such as BC in the figure which extends from the circumference of the circle $(x-a)^2+y^2=a^2$ i. e. $y=\sqrt{2ax-x^2}$ to the line $y=a$. These are therefore the limits of integration w. r. t. y. To have same region of integration as in the given integral, we must take such strips from $x=0$ to $x=a$, which are the limits of integration w. r. t. x. Thus changing the order of integration, the given integral say I, can be written as

$$I=\int_0^a dx\int_{\sqrt{2ax-x^2}}^{a}\frac{xy\log(x+a)}{(x-a)^2}\,dy$$

Integrating w. r. t. y considering x as constant, we have

$$I=\int_0^a dx\,\frac{x\log(x+a)}{(x-a)^2}\left[\frac{y^2}{2}\right]_{\sqrt{2ax-x^2}}^{a}$$

$$=\tfrac{1}{2}\int_0^a dx\,\frac{x\log(x+a)}{(x-a)^2}\left[a^2-2ax+x^2\right]$$

$$=\tfrac{1}{2}\int_0^a x\log(x+a)\,dx.$$

This can be integrated by parts, with log $(x + a)$ as a part to be differentiated, which gives

$$I = \frac{a^2}{8}[\,2 \log a + 1\,]$$

Example 3. Change the order of integration in

$$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)\, dx dy$$

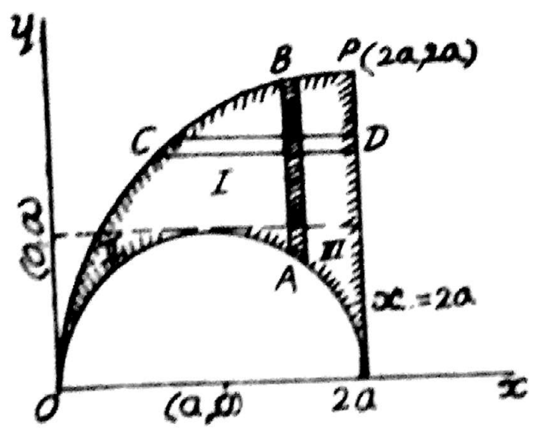


Fig. 177

The order of integration in the given integral is first w. r. t. y and then w. r. t. x.

The elementary strips here are parallel to the y-axis (such as A B) and extend from $y = \sqrt{2ax - x^2}$, [i. e. the circle $x^2 + y^2 - 2ax = 0$. with centre at $(a, 0)$ and radius a] to $y = \sqrt{2ax}$ [i. e. the parabola $y^2 = 2ax$] and such strips are taken from $x = 0$ to $x = 2a$. The shaded area between the parabola and the circle is therefore the region of integration.

In changing the order of integration, we integrate first w. r. t. x, with elementary strips parallel to the x-axis, such as CD. In covering the same region as above, the ends of these strips extend to different curves. We therefore divide the region by the line $y = a$ into three parts (I), (II), (III) as shown above in the figure.

For the region (I), the strips extend from the parabola $y^2 = 2ax$ i. e. $x = \frac{y^2}{2a}$ to the straight line $x = 2a$, so these are the limits of integration w. r. t. x. Such strips are to be taken from $y = a$ to $y = 2a$, to cover the region (I) completely. So the part of the integral in this region I_1 is

$$I_1 = \int_a^{2a} dy \int_{y^2/2a}^{2a} f(x, y)\, dx \quad \dots \quad \dots \quad \text{(i)}$$

For the region (II), the strips extend from the parabola $y^2 = 2ax$ i. e. $x = \frac{y^2}{2a}$ to the circle $x^2 + y^2 - 2ax = 0$ i. e. $x = a \pm \sqrt{a^2 - y^2}$ in which we take the negative sign with the radical as is obvious from the figure, so the

limits of integration w. r. t. x are $x = \frac{y^2}{2a}$ to $x = a - \sqrt{a^2 - y^2}$ and such strips are taken from $y = 0$ to $y = a$, to cover this region completely. The contribution to the integral from this region I_2 is therefore

$$I_2 = \int_0^a dy \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x, y)\, dx \quad \ldots \quad \ldots \quad \text{(ii)}$$

For the region (III), the strips extend from the circle $x^2 + y^2 - 2ax = 0$ [i. e. $x = a \pm \sqrt{a^2 - y^2}$; in this we have to take the positive sign with the radical as is clear from the figure] to the line $x = 2a$, so that the limits of integration w. r. t x are $x = a + \sqrt{a^2 - y^2}$ to $x = 2a$; and such strips are to be taken from $y = 0$ to $y = a$, which covers in the integration the region (III) Denoting this part of integral by I_3, we have

$$I_3 = \int_0^a dy \int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)\, dx \quad \ldots \quad \ldots \quad \text{(iii)}$$

Thus if we change the order of integration, we have to divide the region of integration, and the given integral is equal to $I_1 + I_2 + I_3$ or from (i), (ii), (iii)

$$\int_0^{2a} dx \int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} f(x, y)\, dy = \int_a^{2a} dy \int_{\frac{y^2}{2a}}^{2a} f(x, y)\, dx + \int_0^a dy \int_{\frac{y^2}{2a}}^{a - \sqrt{a^2 - y^2}} f(x, y)\, dx$$

$$+ \int_0^a dy \cdot \int_{a + \sqrt{a^2 - y^2}}^{2a} f(x, y)\, dx.$$

The example illustrates that in changing the order of integration sometimes, not only the limits are to be changed, but it is necessary to split up the region of integration.

Example 4. Change the order of integration for the integral

$$\int_0^a \int_{x^2/a}^{2a - x} xy\, dx\, dy$$

and evaluate the same with reversed order of integration.

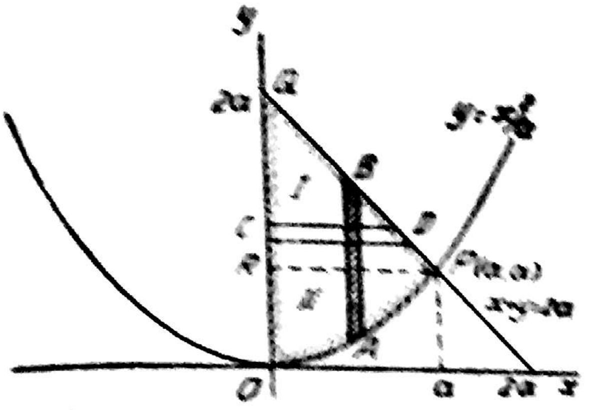


Fig. 178

The given integral is

$$\int_0^a dx \int_{x^2/a}^{2a-x} y\,dy \quad \ldots \quad \ldots \quad \text{(i)}$$

In this the integration is first w. r. t. y, with strips such as AB, parallel to the y-axis with extremities lying on the parabola $y = \dfrac{x^2}{a}$ and the straight line $y = 2a - x$. These strips are taken from $x = 0$ to $x = a$, that gives the region of integration, the curvilinear triangle OPQ, shaded in the above figure.

In changing the order of integration, the integration is to be taken first w. r. t. x, with elementary strip parallel to the x-axis, such as CD, and that needs dividing the region of integration by the line $y = a$, i. e. the line PR, into two parts, the triangle PQR and the curvilinear triangle OPR denoted in the figure by (I) and (II) respectively.

For the region (I), the limits of integration w. r. t. x are $x = 0$ to $x = 2a - y$ and the limits of the next integration w. r. t. y are $y = a$ to $y = 2a$, so the contribution to the given integral from region (I) is

$$I_1 = \int_a^{2a} dy \int_0^{2a-y} xy\,dx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

For the region (II), the limits of integration w. r. t. x are $x = 0$ to $x = \sqrt{ay}$ and those w. r. t. y are $y = 0$ to $y = a$, so the contribution to the given integral from the region (II) is

$$I_2 = \int_0^a dy \int_0^{\sqrt{ay}} xy\,dx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Hence, reversing the order of integration, from (i), (ii) and (iii),

$$\int_0^a dx \int_{x^2/a}^{2a-x} xy\,dy = \int_a^{2a} dy \int_0^{2a-y} xy\,dx + \int_0^a dy \int_0^{\sqrt{ay}} xy\,dx \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

Now, with usual method of evaluating the double integral

$$\int_a^{2a} dy \int_0^{2a-y} xy\,dx = \int_a^{2a} dy \,.\, y \left[\frac{x^2}{2}\right]_0^{2a-y} = \frac{1}{2} \int_a^{2a} y\,(2a-y)^2 dy$$

$$= \frac{5}{24} a^4 \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

and $\int_0^a dy \int_0^{\sqrt{ay}} xy\,dx = \int_0^a dy.\,y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} = \frac{1}{2}\int_0^a ay^2\,dy$

$$= \frac{1}{6}a^4 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(vi)}$$

From (iv), (v) and (vi),

$$\int_0^a dx \int_{x^2/a}^{2a-x} xy\,dy = \frac{5}{24}a^4 + \frac{1}{6}a^4 = \frac{3}{8}a^4.$$

Examples XVI - A.

Evaluate the following integrals :—

1. $\iint y\,dx\,dy$ over (i) the area bounded by $y = x^2$ and $x + y = 2$

(ii) the area bounded by $x = 0$, $y = x^2$, $x + y = 2$ in the first quadrant.

$$\left[\text{Ans. (i) } \frac{36}{5} \quad \text{(ii) } \frac{16}{15}\right].$$

2. $\iint xy(1 - x - y)^{1/2}\,dx\,dy$ over the area of the triangle defined by $x = 0$, $y = 0$ $x + y - 1 = 0$.

$$\left[\text{Ans. } \frac{16}{945}\right].$$

3. (i) $\iint e^{ax+by}\,dx\,dy$ (ii) $\sin \pi(ax + by)\,dx\,dy$ over the area of a triangle bounded by $x = 0$, $y = 0$ and $ax + by = 1$.

$$\left[\text{Ans. (i) } \frac{1}{ab} \quad \text{(ii) } \frac{1}{\pi ab}\right].$$

4. $\iint xy(x + y)\,dx\,dy$ over the area bounded by the parabolas $x^2 = y$ and $y^2 = -x$.

[Ans. 114/420]

5. (i) $\iint (x^2 + y^2)\,dx\,dy$ (ii) $\iint x^2y\,dx\,dy$ over the area in the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\left[\text{Ans. (i) } \frac{\pi ab}{16}(a^2 + b^2) \quad \text{(ii) } \frac{a^4b^2}{24}\right].$$

6. $\iint xy(x + y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$.

[Ans. 3/56].

7. $\displaystyle\iint \frac{dx\,dy}{x^4+y^2}$ where $x > 1$ and $y > x^2$. $\left[\text{Ans. } \dfrac{\pi}{4}\right]$.

8. (i) $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ (ii) $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} x^2y\,dy\,dx$ (iii) $\displaystyle\int_0^1\int_0^{x^2} e^{y/x}\,dy\,dx$.

(iv) $\displaystyle\int_0^{a\sqrt{3}}\left\{\int_0^{\sqrt{x^2+a^2}} \frac{x\,dy}{y^2+(x^2+a^2)}\right\}dx$

$\left[\text{Ans. (i) } \dfrac{\pi}{4}\log(1+\sqrt{2}) \text{ (ii) } a^5/_{15} \text{ (iii) } \dfrac{1}{2} \text{ (iv) } \dfrac{\pi a}{4}\right]$.

Change the order of integration and evaluate :—

9. $\displaystyle\int_0^2\int_0^{x^2/4} xy\,dx\,dy$

10. $\displaystyle\int_0^1\int_x^{2x} dx\,dy$

11. $\displaystyle\int_0^1\int_y^{\sqrt{y}} xy\,dx\,dy$

12. $\displaystyle\int_0^a\int_{x/a}^{\sqrt{x/a}} x^2+y^2)\,dx\,dy$

13. $\displaystyle\int_0^a\int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$

14. $\displaystyle\int_0^a\int_0^{2\sqrt{xa}} x^2\,dx\,dy$

15. $\displaystyle\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$

16. $\displaystyle\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$

17. $\displaystyle\int_0^1 dx\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy$

18. $\displaystyle\int_0^\infty\int_0^\infty e^{-x^2(1+y^2)}x\,dx\,dy$

19. $\displaystyle\int_0^a\int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{(y^4-a^2x^2)}}$.

20. $\displaystyle\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

21. $\displaystyle\int_0^2\int_{2-x}^{2+x} x\,dx\,dy$.

22. $\displaystyle\int_2^8\int_{4/x}^x dy\,dx$

23. $\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dy\,dx$

24. $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$

25. $\int_0^3\int_{4/3}^{\sqrt{25-y^2}} xy\,dx\,dy$

26. $\int_0^{1/4}\int_{x^2}^{\sqrt{x}} dy\,dx$

[Ans. (9) $\frac{1}{3}$ (10) $\frac{1}{2}$ (11) $\frac{1}{24}$ (12) $\frac{a}{4}\left[\frac{a^2}{7}+\frac{1}{5}\right]$ (13) πa

(14) $\frac{\pi a^4}{7}$ (15) $\frac{a^3}{3}\log(1+\sqrt{2})$ (16) $\frac{3a^4}{8}$ (17) $\left[1-\frac{1}{\sqrt{2}}\right]$

(18) $\frac{\pi}{4}$ (19) $\frac{\pi a^2}{6}$ (20) 1 (21) $\frac{16}{3}$ (22) $30-8\log 2$

(23) $\frac{1}{2}$ (24) 5/24. (25) 337/8 (26) 5/64.

Show the region of integration and change the order of integration of the following :—

27. $\int_{-a}^{a}\int_0^{y^2/a} f(x,y)\,dy\,dx$

28. $\int_0^a\int_0^{b\sqrt{\frac{x}{a}}} f(x,y)\,dx\,dy$

29. $\int_1^2\int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} f(x,y)\,dx\,dy$

30. $\int_{-2}^{1}\int_{x^2}^{2-x} f(x,y)\,dx\,dy$

31. $\int_0^b dy\int_{-\sqrt{b^2-y^2}}^{a\sqrt{1-\frac{y^2}{b^2}}} f(x,y)\,dx\,(a>b)$

32. $\int_0^a dy\int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dx$

33 $\int_a^b\int_0^{c^2/x} f(x,y)\,dx\,dy$

34. $\int_0^{a\sqrt{\frac{r^2-b^2}{a^2-b^2}}} dx\int_{\frac{b}{a}\sqrt{a^2-x^2}}^{\sqrt{r^2-x^2}} f(x,y)\,dy,\ (a>r>b)$

35. Prove that $\displaystyle\int_0^a \int_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} dx\,dy = \int_0^b \int_{\frac{a}{b}(b-y)}^{\frac{a}{b}\sqrt{b^2-y^2}} dx dy$. What does this integral represent?

36. $\displaystyle\int_0^{a/2} \int_{x^2/a}^{x-\frac{x^2}{a}} f(x,y)\,dxdy$

37. $\displaystyle\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dx\,dy$

38. $\displaystyle\int_0^{a} \int_{mx}^{lx} f(x,y)\,dx\,dy$

39. $\displaystyle\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$

40. $\displaystyle\int_0^{a} \int_{\frac{b}{a}\sqrt{a^2-x^2}}^{b} f(x,y)\,dx\,dy$

41. $\displaystyle\int_0^{a} \int_{-a+\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\,dy.$

42. $\displaystyle\int_{-a}^{0} \int_{\sqrt{-2ax-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$

43. $\displaystyle\int_{-2a}^{0} \int_{2a-\sqrt{4a^2-y^2}}^{a+\frac{y^2}{4a}} f(x,y)\,dx\,dy$

44. $\displaystyle\int_0^{a} \int_{\sqrt{2ay-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

45. Express the following integrals as single integrals and evaluate

(i) $\displaystyle\int_0^1 \int_0^y (x^2+y^2)\,dx\,dy + \int_1^2 \int_0^{2-y} (x^2+y^2)\,dx\,dy$ $\left[\text{Ans. } \dfrac{4}{3}\right]$

(ii) $\displaystyle\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^2 \int_{x=-1}^{x=1} dx\,dy$ $\left[\text{Ans. } \dfrac{10}{3}\right]$

(iii) $\displaystyle\int_0^{a/\sqrt{2}} \int_0^{x} x\,dx\,dy + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} x\,dx\,dy$ $\left[\text{Ans. } \dfrac{a^3}{3\sqrt{2}}\right]$

16·3. Double integral in Polar co-ordinates :—

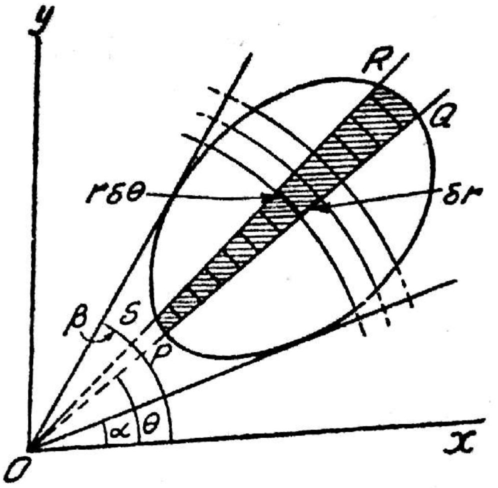


Fig. 179

In case we use polar co-ordinates, divide the region of integration by curves r = const. (which are circles) and θ = const. (which are straight lines).

This gives a mesh of the form shown, where the elementary area is $\delta r . r\,\delta\theta$.

Thus if $f(r, \theta)$ be a function of position, we have over the wedge PQ, the sum as

$$\lim_{\delta r \to 0} \delta\theta \sum_{P}^{Q} f(r, \theta) . r\,\delta r = \delta\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, r dr \quad \ldots \quad \ldots \quad (13)$$

where $r_1(\theta)$ and $r_2(\theta)$ are equations of the two parts of the curves, where θ is kept constant, while integrating w. r. t. r.

Finally summing for all wedges between $\theta = \alpha$ and $\theta = \beta$, we get

$$\lim_{\delta\theta \to 0} \sum_{\alpha}^{\beta} \delta\theta \sum_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, r dr = \int_{\alpha}^{\beta} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, r dr \ldots \quad (14)$$

The order of integration may be changed with appropriate changes in the limits.

Example 1. Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

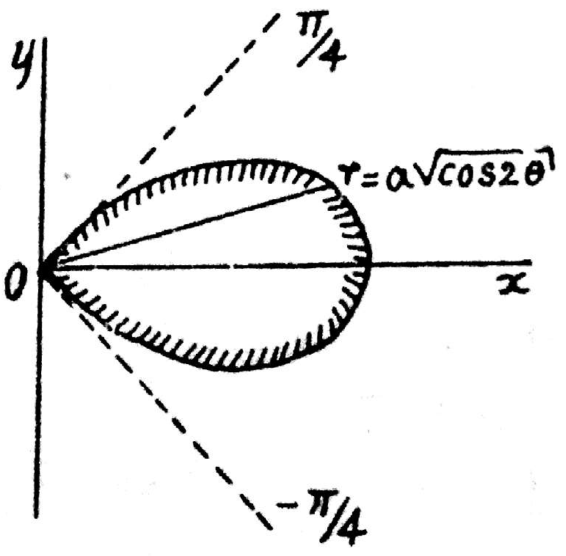


Fig. 180

This curve, as can be known from curve tracing, has a loop lying between $\theta = -\pi/4$ and $\theta = \pi/4$.

If we integrate first w. r. t. r, r has for its limits $r = 0$ and $r = a\sqrt{\cos 2\theta}$ and to cover the entire loop θ varies from $-\pi/4$ to $\pi/4$. Thus the integral required I is

$$I = \int_{-\pi/4}^{\pi/4} d\theta \int_0^{\sqrt{\cos 2\theta}} \frac{r\,dr}{\sqrt{a^2+r^2}}$$

$$= \int_{-\pi/4}^{\pi/4} d\theta \left[\sqrt{a^2+r^2}\right]_0^{a\sqrt{\cos 2\theta}}$$

$$= \int_{-\pi/4}^{\pi/4} d\theta \left\{a\sqrt{1+\cos 2\theta} - a\right\}$$

$$= a\int_{-\pi/4}^{\pi/4} \left[\sqrt{2}\cos\theta - 1\right] d\theta$$

$$= a\left[\sqrt{2}\sin\theta - \theta\right]_{-\pi/4}^{\pi/4}$$

$$= a\left[2 - \frac{\pi}{2}\right] = 2a\left[1 - \frac{\pi}{4}\right].$$

Example 2. Evaluate $\int_0^a dx \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy}{\sqrt{a^2-x^2-y^2}}$, by changing to polar coordinates.

Here the elementary strips, such as AB are parallel to the y axis and extend from $y = \sqrt{ax - x^2}$ [which is the circle $x^2 + y^2 - ax = 0$, with centre at $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$] to $y = \sqrt{a^2 - x^2}$ [i. e. the circle $x^2 + y^2 = a^2$, with centre at the origin and radius a.] Such strips are taken from $x = 0$ to $x = a$, and so the area between the two circles, is the region of integration.

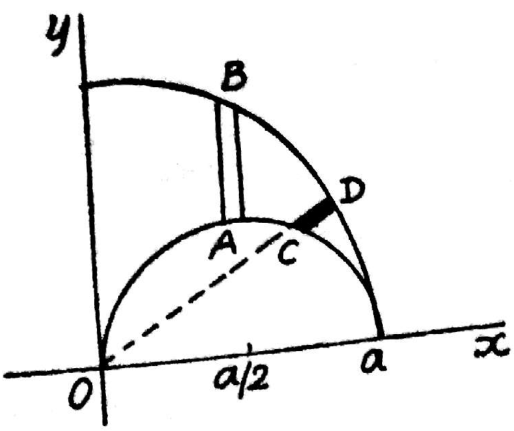


Fig. 181

To change the given integral to polar coordinates, we substitute $x = r\cos\theta$, $y = r\sin\theta$, and $dxdy$ by its equivalent elementary area in polar coordinates $r dr d\theta$. The equations of the circles in polar coordinates are $r = a\cos\theta$ and $r = a$ and the ends of the elementary wedge, such as CD

along theradius-vector lies on these circles and so give the limits of integration w. r. t. r. and to cover the same region of integration as in given integral, θ varies from 0 to $\frac{\pi}{2}$. Thus the transformed integral I is

$$I = \int_0^{\pi/2} d\theta \int_{a\cos\theta}^{a} \frac{r}{\sqrt{a^2 - r^2}}\, dr$$

$$= \int_0^{\pi/2} d\theta \Big[-\sqrt{a^2 - r^2} \Big]_{a\cos\theta}^{a}$$

$$= \int_0^{\pi/2} a \sin\theta \, d\theta = a.$$

Examples XVI B.

Evaluate :—

1. $\iint re^{-r^2/a^2} \cos\theta \sin\theta \, d\theta$ over the upper half of the circle $r = 2a\cos\theta$.

 [Ans. $\frac{a^2}{16}(3 + e^{-4})$].

2. $\iint r dr d\theta$ over the cardioide $r = 1 + \cos\theta$. [Ans. $3\pi/2$].

3. $\iint r^3 dr d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

 [Ans. $\frac{45\pi}{2}$].

4. $\iint r^4 \cos^3\theta \, dr d\theta$ over the interior of the circle $r = 2a\cos\theta$

 [Ans. $\frac{7\pi}{4} a^5$].

Express the following integrals in polar coordinates, showiug the region of integration and evaluate :—

5. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} e^{-x^2 - y^2} dx dy$ [Ans. $\frac{\pi}{4}(1 - e^{-a^2})$]

6. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} y^2 \sqrt{x^2 + y^2} \, dx dy$ [Ans. $\frac{\pi a^5}{20}$]

7. $\displaystyle\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\; dydx$ $\left[\text{Ans. } \dfrac{\pi a^3}{3}\right]$

8. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{2x-x^2}} (x^2+y^2)\; dydx$ $\left[\text{Ans. } \left(\dfrac{3\pi}{8}-\dfrac{2}{3}\right)\right]$

9. $\displaystyle\int_{0}^{a}\int_{y}^{a} \frac{x^2\, dxdy}{\sqrt{x^2+y^2}}$ $\left[\text{Ans. } \dfrac{a^3}{3}\log(1+\sqrt{2})\right]$

10. $\displaystyle\int_{0}^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\; dxdy$ $\left[\text{Ans. } 8a^2\left(\dfrac{\pi}{2}-\dfrac{5}{3}\right)\right]$

11. $\displaystyle\int_{0}^{2}\int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} \frac{dxdy}{(x^2+y^2)^2}$ $[\text{Ans. } \pi]$

12. $\displaystyle\int_{0}^{a}\int_{y}^{a} \frac{x\, dx\, dy}{x^2+y^2}$. $\left[\text{Ans. } \dfrac{\pi a}{4}\right]$

13. $\displaystyle\int_{0}^{a}\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$ $[\text{Ans. } a]$

14. $\displaystyle\int_{0}^{1}\int_{0}^{x} (x+y)\; dydx$ $[\text{Ans. } 1]$.

15. $\displaystyle\int_{0}^{1}\int_{x^2}^{x} (x^2+y^2)^{-1/2}\; dydx$ $[\text{Ans. } (\sqrt{2}-1)]$

16. $\displaystyle\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2+y^2)}\; dydx.$ $[\text{Ans. } \pi/4]$

17. $\displaystyle\int_{0}^{2a}\int_{0}^{\sqrt{2ax-x2}} (x^2+y^2)\quad dxdy$ $\left[\text{Ans. } \dfrac{3\pi a^4}{4}\right]$

18. $\displaystyle\int_0^{2\sqrt{2}} \int_0^{\sqrt{16-x^2}} dydx$ [Ans. $2(\pi + 2)$]

19. $\displaystyle\int_0^4 \int_0^{\sqrt{4x-x2}} \frac{y}{\sqrt{x^2+y^2}} \quad dydx$ $\left[\text{Ans. } \frac{8}{3}\right]$

20. Express as a single integral and evaluate

$$\int_0^{a/\sqrt{2}} \int_0^{x} \cos\{k(x^2+y^2)\}\,dxdy + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} \cos\{k(x^2+y^2)\}\,dxdy.$$

$\left[\text{Ans. } \frac{\pi}{8k} \sin(ka^2)\right]$

Change to polar coordinates and evaluate :—

21. $\displaystyle\iint \frac{x^2-y^2}{(x^2+y^2)^{3/2}}\,dxdy$ over the region of the circle $x^2+y^2 = 2ax$ in the first quadrant. $\left[\text{Ans. } \frac{2a}{3}\right]$

22. $\displaystyle\iint xy\,(x^2+y^2)^{n/2}\,dxdy$ over the positive quadrant of the circle $x^2+y^2 = a^2$, supposing $n+3>0$ $\left[\text{Ans. } \frac{a^{n+4}}{2(n+4)}.\right]$

23. $\displaystyle\iint \frac{x^2y^2\,dxdy}{x^2+y^2}$ over the region included between the circles $x^2+y^2 = a^2$ and $x^2+y^2 = b^2$ $(a>b)$ in the first quadrant. $\left[\text{Ans: } \frac{\pi(a^4-b^4)}{64}\right]$

24. $\displaystyle\iint y^2\,dxdy$ over the area which lies outside the circle $x^2+y^2-ax = 0$ but inside the circle $x^2+y^2-2ax = 0$ $\left[\text{Ans. } \frac{15\pi}{64} a^4\right]$

25. (i) Evaluate $\displaystyle\iint \frac{dxdy}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2+y^2)^2 = x^2-y^2$. $\left[\text{Ans } \frac{\pi-2}{4}\right].$

(ii) Evaluate $\displaystyle\iint \frac{(x^2+y^2)^2}{x^2\,y^2}\,dx\,dy$ over the area common to the ciroleo $x^2+y^2 = ax,\quad x^2+y^2 = by\ (a>b>0)$. [Ans. ab]

26. *Evaluate* :

(i) $\displaystyle\int_0^1 \int_{4y}^4 e^{x^2}\, dx\,dy$

(ii) $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\,dy}{(x^2+y^2+1)^{3/2}}$

(iii) $\displaystyle\int_0^{\infty} dx \int_0^1 \frac{dy}{1+yx^2}$

(iv) $\displaystyle\int_0^1 dx \int_1^{\infty} e^{-y}\, y^x \log y\, dy$

(v) $\displaystyle\int_0^1 dy \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dx$

(vi) $\displaystyle\int_0^a \int_0^x x\sqrt{(a^2-y^2)(x^2-y^2)}\; dx\,dy$

(vii) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \sin\left\{\frac{\pi}{a^2}\,(a^2-x^2-y^2)\right\} dx\,dy$

(viii) $\displaystyle\int_0^a \int_{y^2/a}^{y} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$

(ix) $\displaystyle\int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} \frac{dx\,dy}{(x^2-2x+y-3)^2}$

(x) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dx\,dy}{(1+e^y)\sqrt{1-x^2-y^2}}$

(xi) $\displaystyle\int_0^1 \int_1^{\sqrt{2-y^2}} \frac{y\,dx\,dy}{\sqrt{(2-x^2)(1-x^2y^2)}}$

(xii) $\displaystyle\int_0^a \int_0^x \frac{dx\,dy}{(y+a)\sqrt{(a-x)(x-y)}}$

(xiii) $\displaystyle\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2}\, e^{-(x^2+y^2)}\, dx\,dy$

(xiv) $\displaystyle\int_0^a x\,e^{-x^2}\, dx \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{y}{x^2+y^2}\, e^{-y^2}\, dy$

(xv) $\displaystyle\int_0^{a\sin\beta} \int_{y\cot\beta}^{\sqrt{a^2-y^2}} \log(x^2+y^2)\, dx\,dy \quad (0<\beta<\pi/2)$

(xvi) $\displaystyle\int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x\,dy\,dx}{y^2+(x^2+a^2)}$

Ans. (i) $\frac{1}{8}(e^{4a}-1)$ (ii) 2π (iii) π (iv) e^{-1} (v) $\pi/_4$ (vi) $\frac{8}{45}a^5$

(vii) $\frac{a^2}{2}$ (viii) $\frac{\pi a}{2}$ (ix) $\frac{2}{3}-\frac{1}{2}\log 3$ (x) $\frac{\pi}{2}\log\left(\frac{2e}{1+e}\right)$ (xi) $1-\frac{\pi}{4}$

(xii) $\pi\log 2$ (xiii) e^{-1} (xiv) $\frac{1}{4a^2}[1-(1+a^2)e^{-a^2}]$ (xv) $a^2\beta(\log a-\frac{1}{2})$

(xvi) $\frac{\pi a}{4}$.

Sketch the area of the double idtegration and evaluate :

27. $\displaystyle\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log_e(x^2+y^2)\,dx\,dy\ (a>0).$

$\left[\text{Ans. } \frac{\pi a^2}{4}(\log a-\frac{1}{2})\right].$

28. $\displaystyle\int_0^{\pi/_2} dy \int_0^{y} \cos 2y\sqrt{1-a^2\sin^2 x}\,dx$

[Ans. $\frac{1}{3}a^{-2}\{(1-a^2)^{3/2}-1\}$]

29. $\displaystyle\int_0^1\int_0^{\sqrt{\frac{1}{2}(1-y^2)}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$ $\left[\text{Ans. } \frac{\pi}{4}\right]$

30. $\displaystyle\int_a^{ae^{\pi/_4}} \int_{2\log\left(\frac{r}{a}\right)}^{\pi/_2} r\sqrt{(a^2e^{\theta}-r^2)}\,dr\,d\theta$ $\left[\text{Ans. } \frac{2}{9}a^3\left(e^{3\pi/_2}-1\right)\right].$

31. **Prove that**

(i) $\displaystyle\int_0^a dx \int_{\sqrt{a^2-x^2}}^{\infty} \frac{x^1/_3}{(x^2+y^2)^{3/2}}\,dy = a^{-2/3}\left\{3\sqrt{\pi}\,\frac{\Gamma(^2/_3)}{\Gamma(\frac{1}{6})}-1\right\}$

(ii) $\displaystyle\int_{-1}^{1} \int_0^{1-x} x^{1/3}y^{-1/2}(1-x-y)^{1/2}\,dx\,dy = -\frac{3\pi}{7}$

(iii) $\displaystyle\int_0^{\infty} dx \int_0^1 e^{-x^a y}\,dy = \frac{\Gamma(1/a)}{a-1}\ (a>1)$

16·4. Relation between Beta and Gamma Functions:— We are now in a position to establish the relation between Beta and Gamma functions which were defined previously as

$$\left.\begin{aligned} B(m, n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \\ &= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx \\ \text{and } \Gamma(m) &= \int_0^\infty e^{-x}\,x^{m-1}\,dx. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (15)$$

In the $\Gamma(m)$, if we substitute $x = z^2$, we get

$$\Gamma(m) = 2\int_0^\infty e^{-z^2}\,z^{2m-1}\,dz \quad \ldots \quad \ldots \quad (16)$$

We use here the following result from double integrals.

If $F(x)$ and $\phi(y)$ are functions of x and y only, and the limits of integrations are constants, then the double integral can be represented as a product of two integrals.

Thus

$$\int_a^b\int_c^d F(x)\,\phi(y)\,dx\,dy = \int_a^b F(x)\,dx \,.\int_c^d \phi(y)\,dy \quad \ldots \quad (17)$$

Now by (16) and (17)

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty e^{-x^2}\,x^{2m-1}\,dx \,.\int_0^\infty e^{-y^2}\,y^{2n-1}\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,x^{2m-1}\,y^{2n-1}\,dx dy, \quad \ldots \quad (18)$$

We change this to polar-coordinates with $x = r\cos\theta$, $y = r\sin\theta$, $dxdy \equiv r dr\,d\theta$; the region of integration in this integral is the complete positive quadrant, to cover which, r must be taken from 0 to ∞, and θ from 0 to $\pi/2$; so (18) gives in polar coordinates,

$$\Gamma(m) . \Gamma(n) = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta \, dr d\theta$$

$$= 4 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr . \int_0^{\pi/2} \cos^{2m-1}\theta . \sin^{2n-1}\theta \, d\theta \quad \ldots \quad (19)$$

by (17).

By (16),

$$2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr = \Gamma\,(m+n) \quad \ldots \quad (20)$$

and putting $\sin^2\theta = p$,

$$2 \int_0^{\pi/2} \cos^{2m-1}\theta \quad \sin^{2n-1}\theta \, d\theta = \int_0^1 p^{n-1} \, (1-p)^{m-1} \, dp$$

$$= \mathrm{B}\,(m, n), \text{ by (15).} \quad \ldots \quad (21)$$

Hence from (19), (20), and (21),

$$\Gamma(m) . \Gamma(n) = \Gamma(m+n) . \ \mathrm{B}\,(m, n),$$

or

$$\boxed{\mathrm{B}\,(m, n) = \frac{\Gamma(m) . \Gamma(n)}{\Gamma(m+n)}} \quad \ldots \ldots \quad (22)$$

which gives the relation between Beta and Gamma Functions.

Example. Evaluate $\iint x^{l-1} \; y^{m-1} \, dx dy$ over the triangle given by $x > 0, y > 0, x + y < 1$.

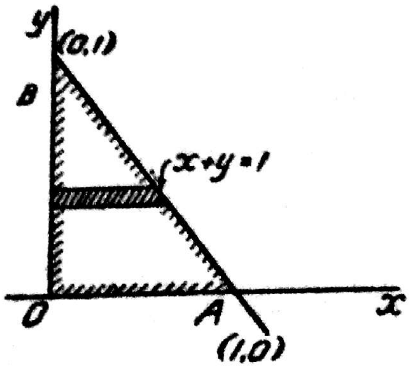


Fig. 182

The region of integration in this case is t e triangle OAB, so the required integral is

$$\int_0^1 dy \int_0^{1-y} x^{l-1} \; y^{m-1} \, dx$$

$$= \int_0^1 y^{m-1} \left[\frac{x^l}{l} \right]_0^{1-y} dy$$

$$= \frac{1}{l} \int_0^1 y^{m-1} \, (1-y)^l \, dy$$

$$= \frac{1}{l} B(m, l+1) = \frac{1}{l} \cdot \frac{\Gamma(m). \Gamma(l+1)}{\Gamma(m+l+1)}$$

$$= \frac{\Gamma(m). \Gamma(l)}{\Gamma(m+l+1)}.$$

16.5. Triple integrals :—

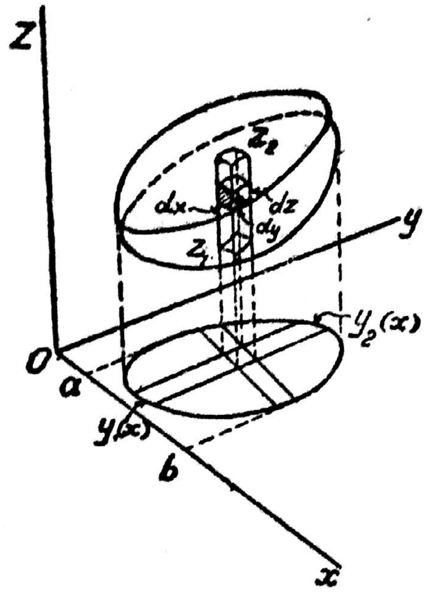


Fig. 183

Let $f(x, y, z)$ be any function of the position of a point (x, y, z) in space [say the density of the body].

Divide the body by a system of planes parallel to the co-ordinate planes into small rectangular blocks. The element of volume at P (x, y, z) is then $dxdydz$.

The mass of the elementary cuboid at P $= f(x, y, z). \; dxdydz$

Then

$$\lim_{dz \to 0} \sum_{z_1}^{z_2} f(x,y,z)\, dxdydz = dxdy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz \quad \dots (23)$$

where $z_1(x, y)$ and $z_2(x, y)$ are the equations of the lower and upper surfaces of the bounding volume. The result (23) gives the mass of the elementary column on $dxdy$ in the xOy plane as the base. In the integral (23), x, y are constants.

We now have to sum for all the columns standing on the area in the xOy plane vertically below the surface. Taking first all the columns in a slice parallel to the y–z plane which means integration w. r. t. y while keeping x constant, we get

$$\left[\int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right\} dy \right] dx \quad \dots \dots (24)$$

and finally summing for all slices from $x = a$ to $x = b$, we have

$$\boxed{\int_v f(x, y, z)\, dv = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz} \quad \dots (25)$$

The evaluation of a space or volume integral involves three successive integrations and so is called a triple integral. The order of integration may be changed with appropriate changes in the limits.

In polar co-ordinates the volume of an elementary cuboid

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

and the integral (25) takes the form

$$\iiint f(r, \theta, \phi) \, r^2 \sin\theta \, dr d\theta d\phi$$

And in cylindrical co-ordinates, the elementary volume is

$$dv = \rho \, d\rho \, d\phi \, dz$$

and the integral (25) takes the from

$$\iiint f(\rho, \phi, z) \, \rho \, d\rho \, d\phi \, dz$$

with appropriate limits.

Example 1. Show that the volume bounded by the cylinder $y^2 = x$, $y = x^2$ and the planes $z = 0$, $x + y + z = 2$ is equal to

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{2-x-y} dx dy dz$$

and evaluate it.

The cylinder stands on the area common to the two parabolas with generators parallel to the z-axis, and the volume required is the portion of this cylinder cut-off by the planes $z=0$ and $x+y+z=2$ i. e. $z=2-x-y$

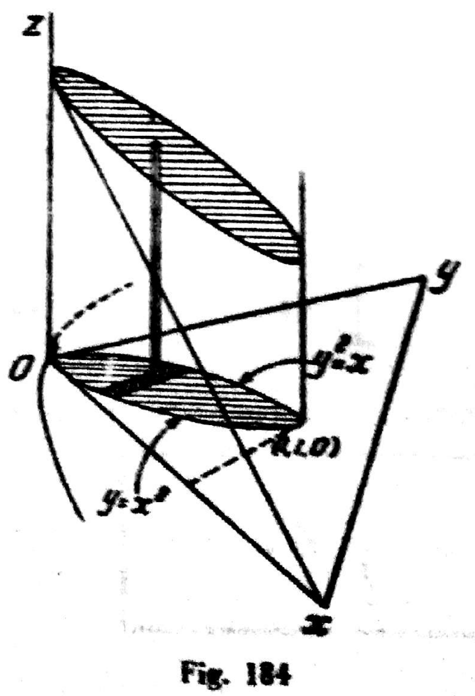


Fig. 184

Integrating first w. r. t. z, we obtain the volume of the elementary column, on $dx\,dy$ as the base, where limits for z are $z = 0$ to $z = 2 - x - y$.

Thus the volume of elementary column on $dxdy$ as the base is

$$dx.dy \int_0^{2-x-y} dz \quad \dots \quad \dots \quad (i)$$

Taking a slice parallel to the yOz plane, of all such columns, leads to an integration w. r. t. y from $y = x^2$ to

$y = \sqrt{x}$ (ref. fig. 184), we thus have the volume of an elementary slice parallel to the yOz plane as

$$dx \int_{x^2}^{\sqrt{x}} dy \int_0^{2-x-y} dz \quad \ldots \ldots \ldots \ldots \quad \text{(ii)}$$

Summing the volumes of such slices, bounded by the curves $y = x^2$, $y = \sqrt{x}$, from $x = 0$ to $x = 1$, gives the total volume of the cylinder in question and is

$$\int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{2-x-y} dz \quad \ldots \ldots \ldots \quad \text{(iii)}$$

which is the same as the given integral. To evaluate it we use the same principles as used in the evaluation of a double integral. Thus

$$\int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{2-x-y} dz = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \Big[z \Big]_0^{2-x-y}$$

$$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} (2-x-y)\, dy$$

$$= \int_0^1 dx \left[(2-x)y - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 \left\{ (2-x)\sqrt{x} - \frac{x}{2} - (2-x)x^2 + \frac{x^4}{2} \right\} dx.$$

$$= \left[\frac{4x^{3/2}}{3} - \frac{2x^{5/2}}{5} - \frac{x^2}{4} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} \right]_0^1$$

$$= \frac{11}{30}.$$

Examples XVI-C.

Evaluate :

1. $\displaystyle\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz.$ $\left[\text{Ans. } \frac{5}{8}\right]$

2. $$\int_0^{2a} dx \int_0^{x} dy \int_y^{x} (xyz)\, dz$$ $\left[\text{Ans. } \frac{4}{3} a^6\right]$

3. $$\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\, dz\,dy\,dx$$ $\left[\text{Ans. } \frac{1}{2}\right]$

4. $$\int_0^{\pi/2} \int_0^{a \sin\theta} \int_0^{\frac{a^2 - r^2}{a}} r\,dr\,d\theta\, dz$$ $\left[\text{Ans. } \frac{5\pi a^3}{64}\right]$

5. $$\int_0^2 \int_x^{4-x} \int_{\frac{3x}{2} - y}^{3} dx\,dy\,dz$$ $\left[\text{Ans. } \frac{43}{3}\right]$

6. $$\int_{-1}^1 \int_0^{z} \int_{x-z}^{x+z} (x + y + z)\, dx\,dy\,dz.$$ $\left[\text{Ans. } 0\right]$

7. $$\int_0^1 \int_{y^2}^{1} \int_0^{1-x} x\, dz\,dx\,dy.$$ $\left[\text{Ans } \frac{4}{35}\right]$

8. $$\int_0^\infty dx \int_0^\infty dy \int_0^\infty \frac{dz}{(1 + x^2 + y^2 + z^2)^2}$$ $\left[\text{Ans. } \frac{\pi^2}{8}\right]$

9. $$\int_0^r \int_0^\alpha \int_0^\beta (x^2 + y^2 + z^2)\, dx\,dy\,dz$$ where $\alpha = \sqrt{r^2 - z^2}$, $\beta = \sqrt{r^2 - y^2 - z^2}$

$\left[\text{Ans. } \frac{\pi r^5}{10}\right]$

and r is a const.

10. $$\int_0^{2a} dx \int_{-\sqrt{2ax - x^2}}^{\sqrt{2ax - x^2}} dy \int_0^{\sqrt{4a^2 - x^2 - y^2}} dz.$$ $\left[\text{Ans. } \frac{8\pi a^3}{3}\right]$

11. Evaluate the integral $\iiint_v \sqrt{x^2 + y^2}\, dx\,dy\,dz$ where v is the volume bounded by the surface $x^2 + y^2 = z^2$, $z > 0$ and the plane $z = 1$.

$\left[\text{Ans : } \frac{5\pi}{3}\right]$

12. Evaluate $\iiint_v (x^2+y^2)\,dx\,dy\,dz$ where v is the volume bounded by the surface $x^2+y^2=2z$ and the plane $z=2$ $\left[\text{Ans. } \dfrac{16\pi}{3}\right]$

13. Change to polar coordinates and evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} \qquad \left[\text{Ans. } \frac{\pi^2}{8}\right]$$

14. Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-y^2-x^2}} (x^2+y^2+z^2)\,dx\,dy\,dz$ $\left[\text{Ans. } \dfrac{\pi a^5}{10}\right]$

15. Show that $\displaystyle\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right)$, integration being taken throughout the volume of the tetrahedran bounded by the coordinate planes and the plane $x+y+z=1$

16. Evaluate $\iiint z^2\,dx\,dy\,dz$ taken throughout the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$.
$\left[\text{Ans. } \dfrac{2\pi}{15}a^5\right]$.

APPLICATIONS OF INTEGRATION

17.1 In this chapter we shall study the applications of integral calculus to the problems involving areas, volumes and surfaces of solids, centre of gravity, hydrostatic centre of pressure, moment of inertia, mean and root mean square values etc. Formulae for these in terms of integrals, single and multiple are developed and their use in the example on these topics is illustrated.

17·2 AREAS

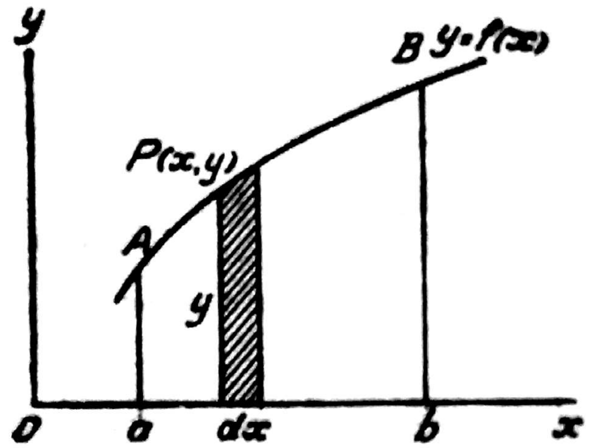


Fig 185

The area A, included by the curve $y = f(x)$ the x-axis and the ordinates $x = a$ and $x = b$ is given by

$$\boxed{A = \int_a^b y\,dx} \qquad \ldots \quad \ldots \quad (1)$$

Similarly the area A′, included by the curve $y = f(x)$, the y-axis, and $y = c$ and $y = d$ is

$$A' = \int_c^d x\,dy \qquad \ldots \qquad \ldots \qquad \ldots \quad (2).$$

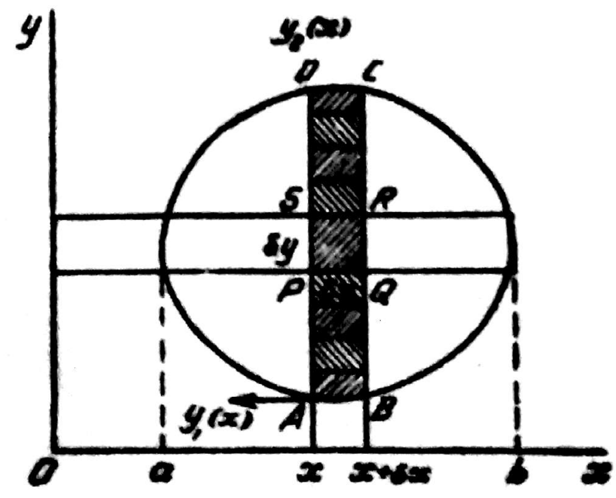


Fig. 186

In case of a loop, as shown in the figure, the area of an elementary rectangle at P (x, y) is $dxdy$ and so the area of the loop is given by

$$\text{Area of the loop} = \int_{a}^{b} \int_{y_1(x)}^{y_2(x)} dx\,dy \quad \ldots \quad (3)$$

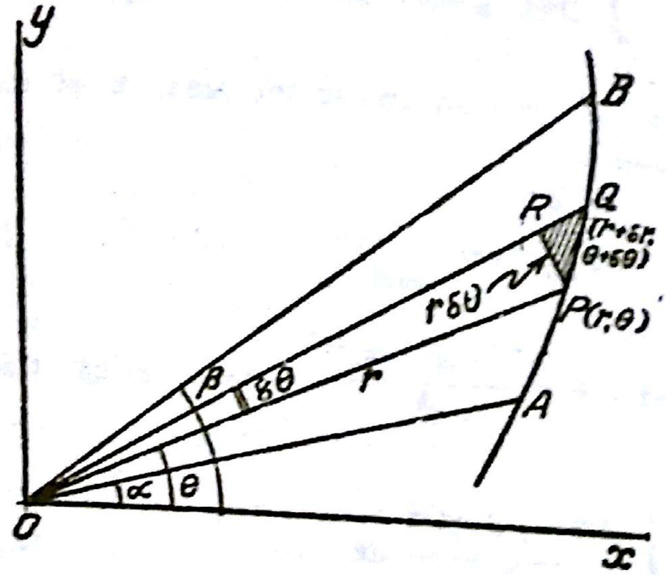


Fig 187

If the equation of curve is given in polar coordinates by $r = f(\theta)$, then as $\delta\theta \to 0$, the area of the elementary triangle OPQ is $\frac{1}{2}r^2\delta\theta$ [for dropping PR perpendicular to OQ, PQ $= r\delta\theta$ can be taken as the base of the $\triangle$ OPR of which the height is r, and its area is $\frac{1}{2}\, r^2\delta\theta$; as $\delta\theta \to 0$, $\triangle$ OPR $\to$ $\triangle$ OPQ.] and so

$$\text{Area of the sector OAB} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta \quad \ldots \ldots \quad (4)$$

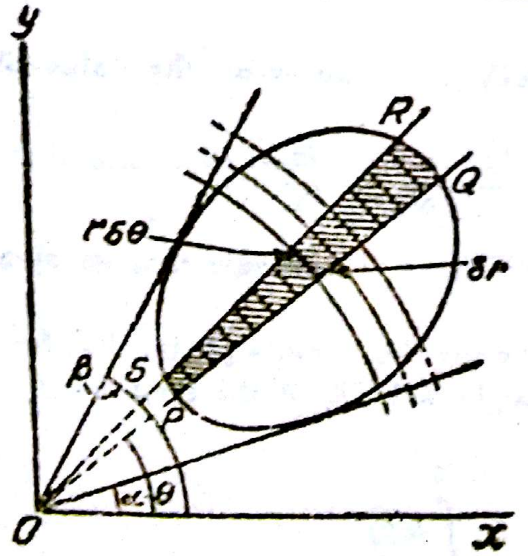


Fig. 188

For a closed curve, whose equation is given in polar co-ordinates, we divide the area into a mesh with lines θ = constant and circles r = constant. The area of an elementary rectangle at P (r, θ) is $r dr d\theta$ and so

$$\text{Area of the loop} = \int \int r\,dr\,d\theta \quad \ldots \ldots \quad (5)$$

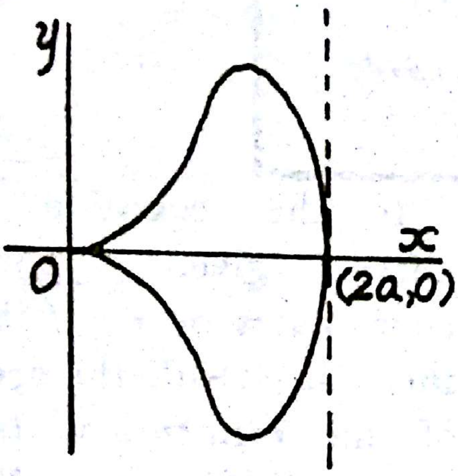


Fig. 189

The tracing done by the methods of curve-tracing (Chap. XIII) gives the curve as a symmetrical *loop on the* x – *axis* between $x = 0$ and $x = 2a$.

$\int_0^{2a} y\,dx$ gives the area of the upper half of the loop and so the area A of the loop is

$$A = 2\int_0^{2a} y\,dx \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

From the equation of the curve $y = \dfrac{x^{5/2}(2a - x)^{1/2}}{a^2}$, substituting this in (i),

$$A = 2\int_0^{2} \frac{x^{5/2}(2a-x)^{1/2}}{a^2}\,dx. \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

For integration, we put $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta\, d\theta$ and when $x = 0,\ \theta = 0$ and when $x = 2a,\ \theta = \dfrac{\pi}{2}$.

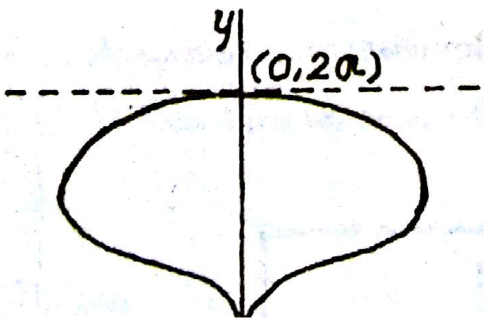
$$\therefore\quad A = 64a^2.\int_0^{\pi/2} \sin^6\theta\cos^2\theta\, d\theta \quad \ldots \quad \ldots \quad \text{(iii)}$$

By the reduction formulae [chapter XIV], we can write the value of this integral, so

$$A = 64a^2.\ \frac{(5.3\cdot1).\ (1)}{8\cdot6.4.2}\ \frac{\pi}{2} = \frac{5\pi}{4}a^2.$$

Example 2. Trace the curve $a^2x^2 = y^3(2a - y)$ and show that its area is equal to πa^2.

Here the *loop is on the* y – *axis*, and so we use the formula (2) for the area. Thus the area A of the loop is



$$A = 2\int_0^{2a} x\,dy$$

$$\int_0^{2a} \frac{y^{3/2}\ (2a- y)^{1/2}}{}$$

$$\text{Substituting } y = 2a\sin^2\theta,\ A = 32\,a^2 \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta$$

$$= 32\,a^2 . \frac{(3.1)\,.\,(1)}{6.4.2.} \frac{\pi}{2} = \pi\, a^2.$$

Example 3. Prove that the area of the loop of the curve $x^5 + y^5 = 5ax^2y^2$ is $\frac{5}{2}a^2$.

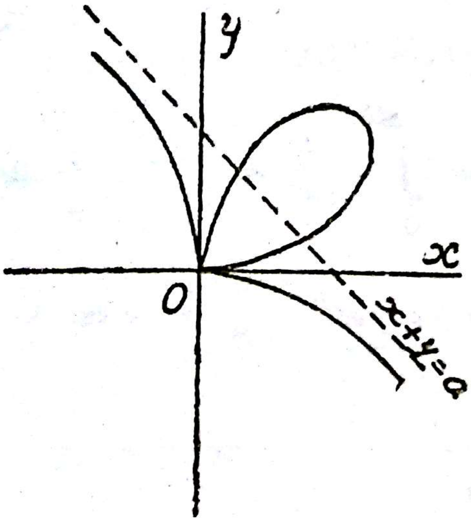


Fig. 191

From the equation of the curve, it is clear that the loop does not lie on the x or y axis and so is inclined to them. In case of *inclined loop*, we change the equation to polar co-ordinates with $x = r\cos\theta$, $y = r\sin\theta$.

The equation of the curve in polar coordinates is

$$r = \frac{5a\sin^2\theta\,\cos^2\theta}{\sin^5\theta + \cos^5\theta} \quad .. \quad .. \quad \text{(i)}$$

r is zero when $\theta = 0$ and $\pi/2$, so the loop of the curve lies between these two limits. Using formula (4), the area A of the loop is

$$A = \frac{1}{2}\int_0^{\pi/2} r^2 d\theta \quad .. \quad .. \quad .. \quad .. \quad \text{(ii)}$$

Substituting for r from (i) in (ii),

$$A = \frac{25a^2}{2}\int_0^{\pi/2} \frac{\sin^4\theta\,\cos^4\theta}{(\sin^5\theta + \cos^5\theta)^2}\, d\theta$$

Dividing the numerator and denominator by $\cos^{10}\theta$,

$$A = \frac{25a^2}{2}\int_0^{\pi/2} \frac{\sec^2\theta\,.\,\tan^4\theta}{(1 + \tan^5\theta)^2}\, d\theta$$

Put $z = 1 + \tan^5\theta$, $dz = 5\sec^2\theta\,\tan^4\theta\, d\theta$. When $\theta = 0$, $z = 1$, and when $\theta = \pi/2$, $z = \infty$,

$$\therefore\ A = \frac{5a^2}{2}\int_1^{\infty} \frac{dz}{z^2} = \frac{5\,a^2}{2}\left[-\frac{1}{z}\right]_1^{\infty} = \frac{5a^2}{2}.$$

Example 4. In the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ find the area between its base and portion of the curve from cusp to cusp.

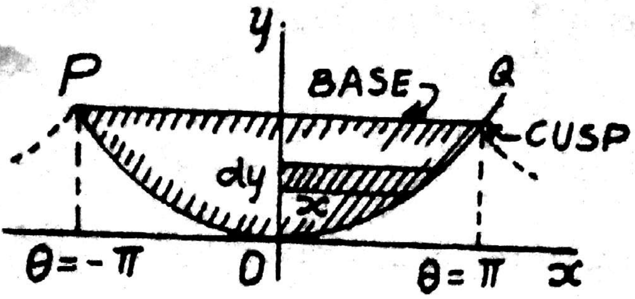


Fig. 192

The sketch of the curve is shown in the figure with cusps at P and Q and the base PQ.

The area required is that of the curviliner figure POQ.

$$\therefore \text{ Required area A} = 2\int x\,dy = 2\int_0^{\pi} x\,\frac{dy}{d\theta}\,d\theta \quad \ldots \quad \ldots \quad \text{(i)}$$

From the equation of the cycloid $x = a(\theta + \sin\theta)$, $\dfrac{dy}{d\theta} = a\sin\theta$

substituting in (i)

$$A = 2\int_0^{\pi} a^2(\theta + \sin\theta)\sin\theta\,d\theta$$

$$= 2a^2\int_0^{\pi}[\theta\sin\theta + \sin^2\theta]\,d\theta \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\text{Now} \int_0^{\pi}\theta\sin\theta\,d\theta = [-\theta\cos\theta + \sin\theta]_0^{\pi} = \pi \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\text{and} \int_0^{\pi}\sin^2\theta\,d\theta = 2\int_0^{\pi/2}\sin^2\theta\,d\theta = 2\cdot\frac{\pi}{4} = \frac{\pi}{2} \quad \ldots \quad \ldots \quad \text{(iv)}$$

Substituting these values of the integrals in (ii).

$$A = 2a^2\left[\pi + \frac{\pi}{2}\right] = 3\pi a^2.$$

Example 5. Find the area between $y^2 = \dfrac{x^3}{a - x}$ and its asymptote.

The nature of the curve is shown in the figure with *asymptote* $x = a$
[Asymptote is the line to which the curve approaches]

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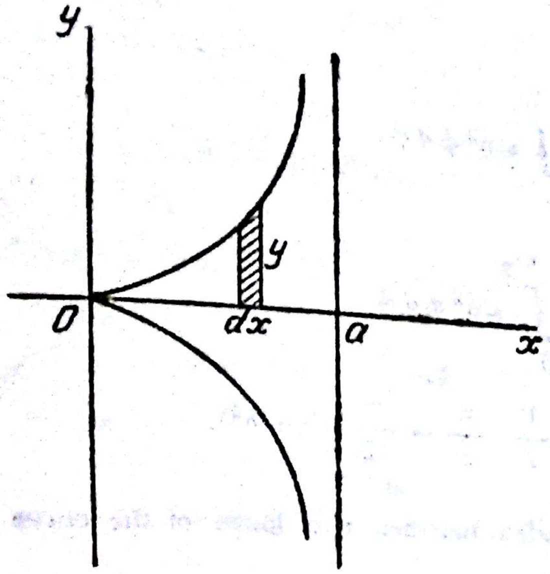

Fig. 193

The required area A is :

$$A = 2\int_0^a y\,dx$$

$$= 2\int_0^a \frac{x^{3/2}}{(a-x)^{1/2}}\,dx \qquad \ldots \quad \text{(i)}$$

with $x = a\sin^2\theta$,

$$A = 4a^2\int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= 4a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{4}\pi a^2.$$

Example 6. Find the area of the loop of the curve

$$r = a\cos 3\theta + b\sin 3\theta.$$

Let $\alpha = \tan^{-1}\dfrac{a}{b}$, so that $a = \sqrt{a^2+b^2}\sin\alpha$, $b = \sqrt{a^2+b^2}\cos\alpha$

so that the equation of the curve can be written as

$$r = \sqrt{a^2+b^2}\,(\sin\alpha\cos 3\theta + \cos\alpha\sin 3\theta).$$

$$\text{or } r = \sqrt{a^2+b^2}\,\sin(3\theta+\alpha) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

To find the position of the loop, we have when $r = 0$, $3\theta + \alpha = n\pi$ (where n is an integer). Taking consecutive values of n as 0 and 1, one of the loop lie between $\theta = -\dfrac{\alpha}{3}$ and $\theta = \dfrac{\pi-\alpha}{3}$.

$$\therefore \text{ The area of the loop} = A = \frac{1}{2}\int_{-\alpha/3}^{\frac{\pi-\alpha}{3}} r^2\,d\theta \qquad \ldots \quad \ldots \quad \text{(ii)}$$

Substituting for r from (i)

$$A = \frac{(a^2+b^2)}{2}\int_{-\alpha/3}^{\frac{\pi-\alpha}{3}} \sin^2(3\theta+\alpha)\,d\theta.$$

In this put $\phi = 3\theta + \alpha$; so that

$$A = \frac{(a^2 + b^2)}{6} \int_0^{\pi} \sin^2 \phi \, d\phi$$

$$= \frac{(a^2 + b^2)}{3} \int_0^{\pi/2} \sin^2 \phi \, d\phi$$

$$= \frac{(a^2 + b^2)}{3} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{12} (a^2 + b^2).$$

Example 7. Find the area included between two loops of the curve

$$r = a(\sqrt{2} \cos\theta - 1).$$

It is necessary to trace this curve properly to get an idea of how one loop lies inside the other. We tabulate the values of r for some convenient values of θ. Since the expression for r involves $\cos\theta$, the curve is symetrical about the initial line, and so a range of values of θ from 0 to π, is sufficient for analysis We have,

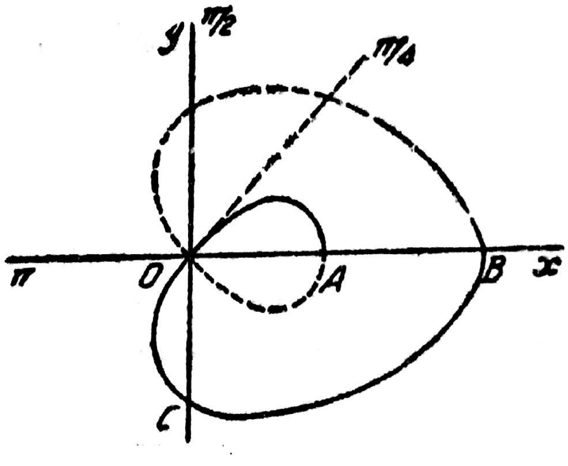

Fig. 191

θ	r
0	$a(\sqrt{2} - 1)$
$\pi/4$	0
$\pi/2$	$-a$
π	$-a(1+\sqrt{2})$

The curve starts at A, where $\theta = 0$, and $r = a(\sqrt{2} - 1)$, r continuously decreases as θ increases, and becomes 0 when $\theta = \pi/4$. As θ increases from $\frac{\pi}{4}$, r is negative, and so is in the opposite direction. Thus for $\theta = \pi/2$, $r = -a$, and so we take OC $= a$ in opposite direction to Oy. At $\theta = \pi$, $r = -a(\sqrt{2} + 1)$, and so intersects Ox at a point .. where OB $= a(1+\sqrt{2})$. We also note that OB > OA. The continuous curve AOCB gives the trace of the curve for θ from 0 to π and by symmetry about the initial line, we get the complete curve.

The half of the inner loop OA is obtained from $\theta = 0$ to $\theta = \pi/4$ and that of the outer loop from $\theta = \pi/4$ to $\theta = \pi$. Thus,

$\therefore$ the area of the inner loop $= A_1 = 2 \cdot \dfrac{1}{2} \displaystyle\int_0^{\pi/4} r^2 \, d\theta$ $\quad \ldots \quad$ (i)

Substituting from the equation of the curve for r,

$$A_1 = a^2 \int_0^{\pi/4} (\sqrt{2} \cos \theta - 1)^2 \, d\theta$$

$$= a^2 \int_0^{\pi/4} [\, 2 \cos^2 \theta - 2\sqrt{2} \cos \theta + 1 \,] \, d\theta$$

$$= a^2 \int_0^{\pi/4} [\, 1 + \cos 2\theta - 2\sqrt{2} \cos \theta + 1 \,] \, d\theta$$

$$= a^2 \left[\frac{1}{2} \sin 2\theta - 2\sqrt{2} \sin \theta + 2\theta \right]_0^{\pi/4} \quad \ldots \quad \text{(ii)}$$

$$= a^2 \left[\frac{1}{2} - 2 + \frac{\pi}{2} \right] = \frac{a^2}{2} (\pi - 3) \quad \ldots \quad \text{(iii)}$$

and the area of the outer loop $= A_2 = 2 \cdot \dfrac{1}{2} \displaystyle\int_{\pi/4}^{\pi} r^2 \, d\theta$ $\quad \ldots \quad$ (iv)

$$\therefore \quad A_2 = a^2 \int_{\pi/4}^{\pi} (\sqrt{2} \cos \theta - 1)^2 \, d\theta$$

$$= a^2 \left[\frac{1}{2} \sin 2\theta - 2\sqrt{2} \sin \theta + 2\theta \right]_{\pi/4}^{\pi}$$

$$= a^2 \left[2\pi - \frac{1}{2} + 2 - \frac{\pi}{2} \right] = \frac{3a^2}{2} (\pi + 1) \quad \ldots \quad \text{(v)}$$

From (iii) and (v),

The area between the two loops $= A_2 - A_1$

$$= \frac{3a^2}{2} (\pi + 1) - \frac{a^2}{2} (\pi - 3)$$

$$= a^2 (3 + \pi).$$

Example 8. Find by double integration the area included between the curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$.

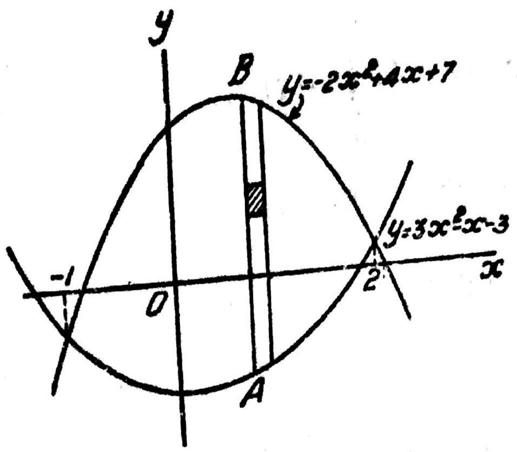

Fig. 195

The abscissa of the points of intersection of the two parabolas, a rough sketch of which is given in the adjacent diagram, are given by

$$3x^2 - x - 3 = -2x^2 + 4x + 7$$

i. e. $x^2 - x - 2 = 0$.

$$\therefore \quad x = -1, 2.$$

Taking the elementary strip parallel to the y – axis, such as AB, bounded by the two parabolas we integrate first w. r. t. y, and then integrating w. r. t. x from $x = -1$ to $x = 2$, gives for the area A required.

$$A = \int_{-1}^{2} \int_{3x^2 - x - 3}^{-2x^2 + 4x + 7} dx\,dy$$

$$= \int_{-1}^{2} dx \left[y \right]_{3x^2 - x - 3}^{-2x^2 + 4x + 7}$$

$$= 5 \int_{-1}^{2} (-x^2 + x + 2)\, dx$$

$$= 5 \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^{2}$$

$$= 5 \left\{ -\frac{8}{3} + \frac{4}{2} + 4 - \left(\frac{1}{3} + \frac{1}{2} - 2 \right) \right\}$$

$$= \frac{5}{2}.$$

Example 9. Find by double integration the area included between the curve $r = a\,(\sec\theta + \cos\theta)$ and its asympotote $r = a \sec\theta$.

By transforming the equations to cartesian coordinates, the curves are easily traced, as shown in the figure.

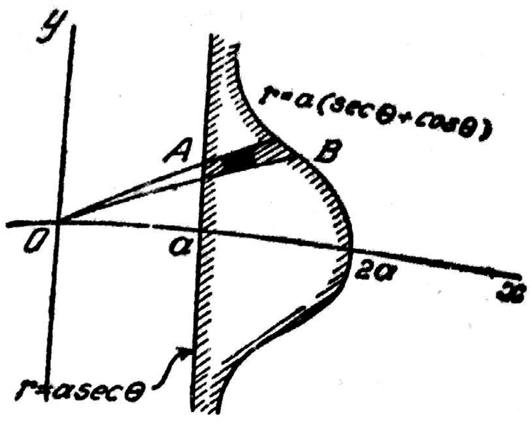

Fig. 196

Taking a wedge such as AB, its extremities lie on the curve $r = a \sec\theta$ and $r = a\,(\sec\theta + \cos\theta)$ and to get the area between the asymptote and the curve, θ varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$; or by symmetry the area A required is :

$$A = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\Big[r^2\Big]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}\{(\sec\theta + \cos\theta)^2 - \sec^2\theta\}\,d\theta$$

$$= a^2\int_0^{\pi/2}[\,2 + \cos^2\theta]\,d\theta$$

$$= a^2\left[\pi + \frac{\pi}{4}\right] = \frac{5\pi}{4}\,a^2.$$

Examples : XVII—A

1. Obtain the area in the first quadrant bounded by the curve $b^4y^2 = (a^2 - x^2)^3$ and the co-ordinate axes. $\left[\text{Ans. } \dfrac{3\pi}{16}\dfrac{a^4}{b^2}\right]$.

2. An ellipse of small eccentricity has its perimeter equal to that of a circle of radius a. Show that its area is $\pi a^2\left(1 + \dfrac{3}{32}e^4\right)$ nearly.

3. Prove that the area of the curve $x = \dfrac{a}{(1 + t + t^2)}$, $y = \dfrac{bt}{(1 + t + t^2)}$ is $\dfrac{2}{9}\pi ab\sqrt{3}$.

4. Find the area enclosed by the curves $x^2 = 4ay$ and $x^2 + 4a^2 = \dfrac{8a^3}{y}$

$\left[\text{Ans. } \dfrac{2a^2}{3}(3\pi - 2)\right]$.

5. Find the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$. $\left[\text{Ans. } \frac{3\pi}{8} a^2\right]$.

6. Find the area of the loop of the following curves.

(i) $ay^2 = x(x-a)^2$. (ii) $(x+y)(x^2+y^2) = 2axy$.

(iii) $x^3 + y^3 = 3axy \ (a > 0)$. (iv) $y^2 = \frac{x^2}{a^2}(2a-x)(x-a)$.

(v) $x^2(x^2+y^2) = a^2(x^2-y^2)$. (vi) $y^2 = (x-a)(b-x), 0 < a < b$

(vii) $x^4 + y^4 = 2a^2xy$. (viii) $x^4 + y^4 = a^2x^2y^2$

$\left[\text{Ans. (i) } \frac{8}{15} a^2. \text{ (ii) } a^2\left(1 - \frac{\pi}{4}\right). \text{ (iii) } \frac{3}{2} a^2. \text{ (iv) } \frac{3\pi}{8} a^2.\right.$

$\left.\text{(v) } \frac{1}{2}(4-\pi)a^2. \text{ (vi) } \frac{\pi}{4}(a-b)^2. \text{ (vii) } \frac{\pi a^2}{4}. \text{ (viii) } \pi a^2/_{12}.\right]$

7. Show that the area between the curves

$$y = ax^2 \text{ and } y = 1 - \frac{x^2}{a} \ (a > 0) \text{ is } \frac{4}{3}\sqrt{\frac{a}{a^2+1}}.$$

8. Find the area enclosed in the first quadrant between the curve $6xy = x^4 + 3$ and the line $3y = 2x$ and show that the length of the arc cut off by the line is $\frac{1+\sqrt{3}}{3}$. $\left[\text{Ans. } \frac{1}{3} - \frac{1}{4}\log 3\right]$.

9. Find in tems of Gama functions, the area enclosed by the curve

$$\left(\frac{x}{a}\right)^4 + \left(\frac{y}{b}\right)^{10} = 1$$

10. Show that if s be the arc of the curve $r = a \sec \alpha$, $\theta = \tan \alpha - \alpha$ where α is a variable parameter, measured from the initial line to a point P in the curve and if A be the area bounded by the curve the initial line and the radius vector to P, then $9A^2 = 2as^3$.

11. Sketch the curve whose polar equation is

$$r = a \tanh \frac{\theta}{2}.$$

Find the length s of the curve and the area A of the sector measured from the radius vector at $\theta = 0$ to the radius vector (r, θ) and show that $2A = a(s - r)$.

12. If A is the vertex, O the centre and P (x, y) any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, prove that $x = a\cosh \frac{2S}{ab}$, $y = b \sinh \frac{2S}{ab}$, where S is sectorial area OPA.

13. Find the total area of the curve $y^2 = x^2 \cdot \frac{a^2 - x^2}{a^2 + x^2}$.

[Ans. $a^2(\pi - 2)$]

14. Find the area of the loop of the curve $x(x^2+y^2) = a(x^2-y^2)$ and also the area enclosed by the curve and its asymptote.

$$\left[\text{Ans. } a^2\left(2-\frac{\pi}{2}\right), a^2\left(2+\frac{\pi}{2}\right)\right]$$

15. Find the whole area included between the curve $x^2y^2 = a^2(y^2-x^2)$ and its asymptote. [Ans. $4a^2$]

16. Show that the area of loop of the curve $y^2(a+x) = x^2(3a-x)$ is equal to the area between the curve and its asymptote.

17. Find the area between the curve $(a-x)y^2 = a^2x$ and its asymptote. [Ans. πa^2].

18. Find by double integration, the area between the curve $y = x^2 - 6x + 3$ and $y = 2x - 9$.

$$\left[\text{Ans. } 10\frac{2}{3}\right].$$

19. Show by double integration, that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

20. Find by double integration, the area between the curve

$$y^2 = \frac{4a^2(2a-x)}{x} \text{ and its asymptote.}$$ [$4\pi a^2$]

21. Find the area between the curve $y^2 = 4x$ and $2x - 3y + 4 = 0$.

$$\left[\text{Ans. } \frac{1}{3}\right]$$

22. Find the area included between the curves $9xy = 4$ and $2x + y = 2$ by double integration.

$$\left[\text{Ans. } \frac{1}{3} - \frac{4}{9}\log 2\right]$$

23. Show that the area enclosed by the curves $xy^2 = a^2(a-x)$ and $(a-x)y^2 = a^2x$ is $(\pi - 2)a^2$.

24. Find the area common to the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

$$\left[\text{Ans. } 4ab\tan^{-1}\frac{b}{a}\right]$$

25. Find by double integration the area included between the curves $y^2 = 4a(x+a)$ and $y^2 = 4b(b-x)$.

$$\left[\text{Ans. } \frac{8}{3}(a+b)\sqrt{ab}\right]$$

26. Prove that the area of the part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ which is within the parabola $b^2x^2 = (a^2-b^2)ay$ is given by $\frac{1}{3}b^2e + \sin^{-1}e$, where e is the eccentricity of the ellipse.

27. Express as a repeated integral the area of the curvilinear triangle, lying in the first quadrant with one vertex at the origin and bounded by the curves $y^2 = 8x$, $x^2 = 8y$ and $x^2 + y^2 = 20$.
Also evaluate the area. $\left[\text{Ans. } \frac{8}{3} + 10\left(\sin^{-1}\frac{2}{\sqrt{5}} - \sin^{-1}\frac{1}{\sqrt{5}}\right)\right]$.

28. Find the area of the closed portion of the Folium
$$r = \frac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$$
$\left[\text{Ans. } \frac{3a^2}{2}\right]$.

29. Show that the area of a loop of the curve $r = a\cos n\theta$ is $\frac{\pi a^2}{4n}$ and state the total area in cases n is odd, n is even. Also find the area contained between the circle $r = a$ and the curve $r = a\cos 5\theta$.
$\left[\text{Ans. } n\text{-odd } \frac{\pi a^2}{4};\ n\text{-even } \frac{\pi a^2}{2};\ \frac{3}{4}\pi a^2\right]$.

30. Find the area included between two loops of the curve
$r = a(2\cos\theta + \sqrt{3})$. $\left[\text{Ans. } \frac{a^2}{3}(10\pi + 9\sqrt{3})\right]$.

31. Prove that the curve $r = a\left(\frac{\sqrt{3}}{2} + \cos\frac{\theta}{2}\right)$ has three loop whose areas are $a^2\left(\frac{5\pi}{4} + 2\sqrt{3}\right)$, $a^2\left(\frac{5\pi}{6} - \frac{5\sqrt{3}}{4}\right)$, $a^2\left(\frac{5\pi}{12} - \frac{3\sqrt{3}}{4}\right)$ respectively.

32. Find the areas of three loops of the curve $r = a\left(1 + 2\sin\frac{\theta}{2}\right)$.
$\left[\text{Ans. } (3\pi + 8)a^2,\ (3\pi - 8)\frac{a^2}{2},\ (3\pi - 8)\frac{a^2}{2}\right]$

33. Find the area bounded by the curve $r = 2a\cos 3\theta$ and lying outside the circle $r = a$.
$\left[\text{Ans. } a^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\right]$

34. Find the whole area of the curve represented by the equation
$r = a + b\cos\theta$, assuming $a > b$. $\left[\text{Ans. } \pi\left(a^2 + \frac{1}{2}b^2\right)\right]$.

35. Show that the area of loop of the curve
(i) $r\cos\theta = a\cos 2\theta$ is $\frac{a^2(4 - \pi)}{2}$.
(ii) $r = a\theta\cos\theta$ is $\frac{\pi a^2}{96}(\pi^2 - 6)$

36. Show that the area of the loop of the curve
$r^2(2c^2\cos\theta - 2ac\sin\theta\cos\theta + a^2\sin^2\theta) = a^2c^2$ is πac.

37. Find the area of the curve $r^2 = a^2\cos 2\theta$. [Ans. a^2]

38. Show that the area bounded by the spiral $r = ae^{k\theta}$ and two radii is proportional to the difference of the squares of these radii.

39. Find the area common to the cardioides $r = a(1 - \cos\theta)$, $r = a(1 + \cos\theta)$.

$$\left[\text{Ans. } 2a^2\left(\frac{3\pi}{4} - 2\right)\right].$$

40. Find the area inside the cardioide $r = 2a(1 + \cos\theta)$ and outside the parabola $r = \dfrac{2a}{1 + \cos\theta}$.

$$\left[\text{Ans. } a^2\left(3\pi + \frac{16}{3}\right)\right].$$

41. Show that the area in the first quadrant bounded by the curve

$$x^{2m} + y^{2n} = 1 \text{ is } \frac{\Gamma\left(\frac{1}{2m}\right)\cdot\Gamma\left(\frac{1}{2n}\right)}{4mn\,\Gamma\left(\frac{1}{2m} + \frac{1}{2n} + 1\right)}.$$

17·4 VOLUMES OF SOLIDS

Let $z = f(x, y)$ be the equation of the surface, of which the orthogonal projection in the xOy plane is the

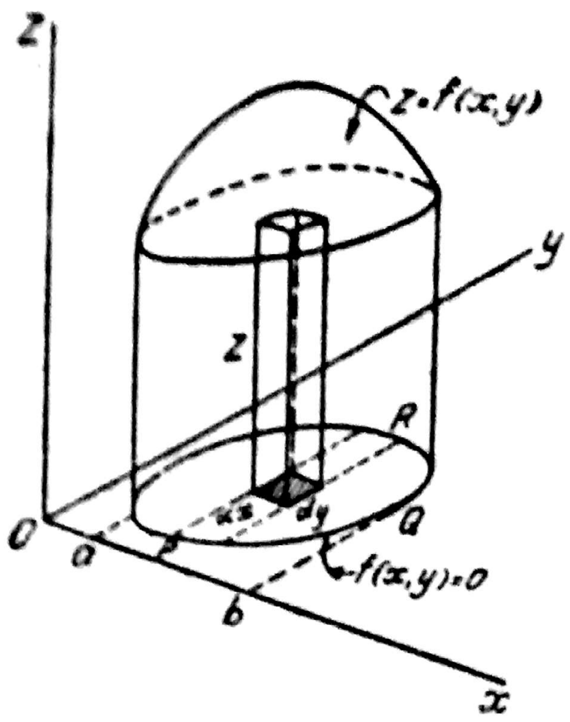

Fig. 199

contour PQR, whose equation is $f(x, y) = 0$. The volume of an elementary parallellopiped on $dxdy$ bounded by the surface, $z = f(x, y)$ and sides parallel to the z axis is $z\,dxdy = f(x, y)\,dx\,dy$.

The summation of all such terms over the area of closed curve PQR gives the volume of the solid cylinder bounded by the given surface and the plane xOy with generators parallel to the z-axis as

$$\boxed{\text{Volume} = \iint f(x, y)\,dx\,dy} \qquad \ldots \quad \ldots \quad (8)$$

to be taken on the area of the contour PQR.

To express the volume of a solid as a triple integral, we note that the volume of an elementary cuboid is $dx\,dy\,dz$; and so the volume of the solid is given by

$$\boxed{\text{volume} = \iiint dx\,dy\,dz} \qquad \ldots \quad \ldots \quad (9)$$

where the limits of integration w. r. t. z (if we integrate first w. r. t. z) are z_1 and z_2 obtained from its equations to the top and bottom of the given surface and then the double integration is w. r. t. x and y is performed over the area of projection of the given solid on the xOy plane.

If $\rho = f(x, y, z)$ is the density of the solid at the point P (x, y, z), then the mass of the solid is

$$\iiint f(x, y, z)\,dx\,dy\,dz \qquad \ldots \quad \ldots \quad \ldots \quad (10)$$

with appropriate limits of integrations.

Example 1. Find by double integration the volume of the sphere $x^2+y^2+z^2=a^2$ cut off by the plane $z=0$ and the cylinder $x^2+y^2=ax$.

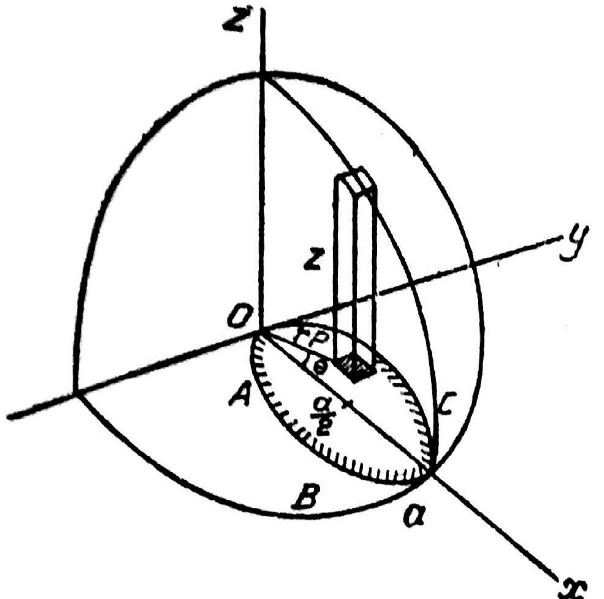

Fig. 200

Taking the polar co-ordinate in the xOy plane, elementary area at P (r, θ) is $r dr d\theta$. If the line at P drawn parallel to the $z-$ axis has length z, the volume of the elementary parallelopiped at P is $z r dr d\theta$, and the volume of the cylinder on the circle $x^2+y^2=ax, z=0$ bounded at the top by the surface of the sphere

$$x^2+y^2+z^2=a^2 \text{ is}$$

$$\iint z\, r\, dr\, d\theta \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

with proper limits of integration.

As $x^2+y^2=r^2$ so the equation of the sphere is $z^2+r^2=a^2$ or $z=\sqrt{a^2-r^2}$. The region of integration is the circle $x^2+y^2-ax=0$ which has its centre at $\left(\frac{a}{2}, 0, 0\right)$ and radius is $\frac{a}{2}$. Its polar equation is $r=a\cos\theta$. So the limits of integration w. r. t. r are 0 and $a\cos\theta$ and w. r. t. θ are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. With these considerations and using (i), the volume V required is (by Symmetry)

$$V = 2\int_0^{\pi/2} d\theta \int_0^{a\cos\theta} \sqrt{a^2-r^2}\, r\, dr \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

To evaluate the first integral put $t^2=a^2-r^2$, so we have

$$\int_0^{a\cos\theta} \sqrt{a^2-r^2}\; r dr = -\int_a^{a\sin\theta} t^2\, dt = -\left[\frac{t^3}{3}\right]_a^{a\sin\theta}$$

$$= \frac{1}{3} a^3 [\,1-\sin^3\theta\,] \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Using this in (ii), the volume required is

$$V = \frac{2a^3}{3}\int_0^{\pi/2} [\,1-\sin^3\theta\,]\, d\theta = \frac{a^3}{1}(3\pi-4)$$

Example 2. Find the volume cut off from the paraboloid

$$x^2+\frac{1}{4}y^2+z=1 \text{ by the plane } z=0.$$

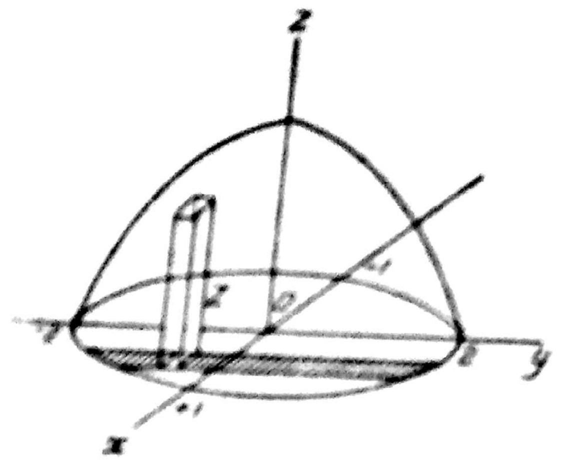

Fig. 201

The equation of the paraboloid is $z = 1 - x^2 - \frac{1}{4}y^2$ and it cuts the plane $z = 0$ in the ellipse $x^2 + \frac{1}{4}y^2 = 1$.

Hence the volume of the paraboloid above the xOy plane

$$= \iint (1 - x^2 - \tfrac{1}{4}y^2)\,dx\,dy$$

taken over the area of the ellipse.

The Volume required

$$= \int_{-1}^{1} dx \int_{-2\sqrt{1-x^2}}^{+2\sqrt{1-x^2}} (1 - x^2 - \tfrac{1}{4}y^2)\,dy$$

$$= \int_{-1}^{1} \left\{ (1-x^2)\,y - \frac{y^3}{12} \right\}_{-2\sqrt{1-x^2}}^{+2\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \frac{8}{3}(1-x^2)^{3/2}\,dx$$

$$= \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta, \text{ (where } x = \sin\theta \text{)}$$

$$= \frac{8}{3} \cdot 2\left[\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right] = \pi.$$

Examples : XVII—C

1. Find the volume bounded by the cylinders $y^2 = x$ and $x^2 = y$, and the planes $z = 0$ and $x + y + z = 2$. $\left[\text{Ans. } \frac{11}{30}\right]$

2. A cylindrical hole of radius b is bored symmetrically through a sphere of radius a. Show that the volume of the remaining solid is

$$2\int_{0}^{2\pi}\int_{b}^{a} \sqrt{a^2 - r^2}\; r\,dr\,d\theta$$

and evaluate it. $\left[\text{Ans. } \frac{4}{3}\pi(a^2 - b^2)^{3/2}\right]$

3. If the density at a point varies as the square of the distance of the point from the xy – plane, find the mass of the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and cylinder $x^2 + y^2 = ax$.

$$\left[\text{Ans. } \frac{2a^5}{15}\left(\pi - \frac{16}{15}\right).\right]$$

4. Find the volume bounded by the surface $z = c\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$ and the positive quadrant of the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$.

$$\left[\text{Ans. } \frac{abc}{4}\left(\pi - \frac{13}{6}\right)\right].$$

5. Find the volume in the first octant bounded by the cylinder $(x - 1)^2 + (y - 1)^2 = 1$ and the paraboloid $xy = z$ [Ans. π].

6. Find the volume under the spherical surface $x^2 + y^2 + z^2 = a^2$ and over the leniniscate $r^2 = a^2 \cos 2\theta$. $\left[\frac{\pi a^3}{3}(1+2\sqrt{2}) + \frac{4}{9} a^3 (2\sqrt{2}-1)\right]$

7. Find the volume of the solid bounded by the surfaces $z = 4 - x^2 - \frac{1}{4}y^2$ and $z = 3x^2 + \frac{1}{4}y^2$ [Ans. $4\sqrt{2}\,\pi$]

8. Find the volume in the first octant bounded by $x^2 + y^2 = 2$, $z = x + y$, $y = x$, $z = 0$ and $x = 0$ $\left[\text{Ans. } \frac{1}{8}(\pi + 1)\right]$

9. A solid is cut out of the cylinder $x^2 + y^2 = a^2$ by the plane $z = 0$ and that part of the plane $z = mx$ for which z is positive the density of the solid at any point varies as the height of the point above the plane $z = 0$. Find the mass of the solid.

$$\left[\text{Ans. } \frac{km^2a^4\pi}{16}\right].$$

10. A solid spheroid is formed by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis; a cylindrical hole of circular cross-section with major axis as its axis is drilled through the solid. Prove that the volume of the remaining solid is $\frac{4\pi b^2 l^3}{3a^2}$, where $2l$ is the length of the hole.

11. Find the volume common to the right circular cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$. $\left[\text{Ans. } \frac{16}{3}a^3\right]$

12. A conical tent of height h is set up over a base which is a cardioid $r = a(1 + \cos\theta)$. The tent pole stands vertically on the origin. Obtain

the expression in polar form of a double integral for the volume within the tent and evaluate it.

$$\left[\text{Ans. } 2h \int_0^{\pi} d\theta \int_0^{a(1+\cos\theta)} \left\{1 - \frac{r}{a(1+\cos\theta)}\right\} r\,dr = \frac{1}{2}\pi a^2 h\right]$$

13. A right circular cylinder of radius $\frac{a}{2}$ and height a is formed by the plane $z = 0$, $z = a$ and the surface $x^2 + y^2 = ax$. Find the volume of the portion of the cylinder inside the cone $x^2 + y^2 = z^2$.

$$\left[\text{Ans. } \frac{a^3}{36}(9\pi - 16)\right]$$

14. The base of pile of sand covers the region in the $x - y$ plane, that is bounded by the parabola $x^2 + y = 6$ and the line $y = x$. The depth of the sand above the point (x, y) is x^2. Find the volume of the sand. [Ans. 87.95]

CHAPTER 14

REDUCTION FORMULAE

14·1. In many problems of integration, we see that by some convenient substitution the integrals can be reduced to the form $\int \sin^n x\, dx$, $\int \cos^n x\, dx$, or $\int \sin^m x \cos^n x\, dx$ and these can be made to depend upon those involving lower degrees of the trigonometrical expressions, by what are known as reduction formulae. Also we shall see that the knowledge of the values of the definite integrals such as $\int_0^{\pi/2} \sin^n x\, dx$ etc. is very helpful in problems of integration. We shall therefore investigate the reduction formulae for these integrals.

14·2. Reduction formula for $\int \sin^n x\, dx$:—

Let $I_n = \int \sin^n x\, dx$ where n is a positive integer.

We shall see that it can be made to depend on $I_{n-2} = \int \sin^{n-2} x\, dx$. Thus :

$$\int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx.$$

Integrating by parts,

$$= -\sin^{n-1} x \cos x$$
$$+ (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

In the integral write $\cos^2 x = 1 - \sin^2 x$, and so we have

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x$$
$$+ (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

or $I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2} - (n-1)\, I_n.$

Transposing the last term on r. h. s. to the l. h. s. we have
$n\,I_n = -\sin^{n-1} x \cos x + (n-1)\,I_{n-2}$

$$\text{or } I_n = -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2} \quad \ldots \quad \ldots \quad (1)$$

This is the required reduction formula, which makes $\int \sin^n x\, dx$ depend upon the evaluation of $\int \sin^{n-2} x\, dx$.

The successive application of this formula makes I_n depend upon I_{n-2}, I_{n-2} upon I_{n-4} and so on and thus depending upon whether n is even or odd, we finally come to the integral

$$I_0 = \int \sin^0 x\, dx = \int dx = x, \text{ when } n \text{ even.}$$

$$\text{or } I_1 = \int \sin x\, dx = -\cos x, \text{ when } n \text{ odd.}$$

14.3. Reduction Formula for $\int_0^{\pi/2} \sin^n x\, dx$:—

We shall now consider the corresponding definite integral.

Let $S_n = \int_0^{\pi/2} \sin^n x\, dx$. Then by (1) above

$$S_n = \int_0^{\pi/2} \sin^n x\, dx = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\therefore \quad S_n = \frac{n-1}{n} S_{n-2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

By successive application of this formula, we have

$$S_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} S_{n-4}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} S_{n-6} \text{ and so on.}$$

The integral finally depends upon

$$S_0 = \int_0^{\pi/2} dx = \frac{\pi}{2} \quad \ldots \text{ if } n \text{ is even}$$

$$\text{or} \quad S_1 = \int_0^{\pi/2} \sin x \, dx = \Big[-\cos x \Big]_0^{\pi/2} = 1 \ldots \text{ if } n \text{ is odd.}$$

We thus have the value of the definite integral as

$$\boxed{\left.\begin{aligned} \int_0^{\pi/2} \sin^n x \, dx &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \; n \text{ even} \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots \frac{2}{3} \cdot 1, \; n \text{ odd.} \end{aligned}\right\}} \quad (3)$$

14·4 Reduction Formula for

$$\int \cos^n x \, dx \text{ and } \int_0^{\pi/2} \cos^n x \, dx :-$$

As above, if $I_n = \int \cos^n x \, dx$ then splitting $\cos^n x$ as $\cos^{n-1} x \cos x$ and integrating by parts, we obtain the reduction formula for $\int \cos^n x \, dx$ as

$$\int \cos^n x dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\text{or } I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

and putting the limits as 0 and $\pi/2$, we have the reduction formula for $C_n = \int_0^{\pi/2} \cos^n x \, dx$ as

$$C_n = \frac{n-1}{n} C_{n-2} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Using this formula, we get the value of the definite integral as

$$\left.\begin{aligned}\int_0^{\pi/2} \cos^n x\,dx &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, \; n \text{ even}\\ &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}\cdot 1, \; n \text{ odd}\end{aligned}\right\} \quad (6)$$

Incidentally it may be noted that $S_n = C_n$ [Ref. next art.]

14.5. Useful Theorems on Definite Integrals :—

These Theorems are useful when the range for integrals of $\sin^n x$, $\cos^n x$ and $\sin^m x \cos^n x$, is multiple of $\frac{\pi}{2}$ i. e. 0 to $\pi/2$ — π to π and 0 to 2π.

Theorem I :—

$$\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx} \quad \ldots \quad \ldots \quad (A)$$

Let $x = a - t \quad \therefore \quad dx = -dt$

$$\int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt$$

$$= \int_0^a f(a-x)\,dx$$

[For definite integral, the substitution of x for t does not change the value of the integral.]

Ex. Prove that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

By the above theorem,

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \cos^n x\,dx$$

Theorem II : –

$$\boxed{\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx} \quad \ldots \quad (B)$$

Now

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \ldots \quad \ldots \quad (i)$$

For the first integral on the R. H. S. put $x = -t$

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt$$

$$= \int_{0}^{a} f(-x)\,dx$$

Substituting in (i), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx$$

Ex. 1. Evaluate $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^2 x\,dx$.

$$\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^2 x\,dx = \int_{0}^{\pi/2} \sin^4 x \cos^2 x\,dx$$

$$+ \int_{0}^{\pi/2} \sin^4 (-x) \cos^2 (-x)\,dx$$

[by the theorem II]

$$= 2 \int_{0}^{\pi/2} \sin^4 x \cos^2 x\,dx \quad \text{[Ref. art. 14·7]}$$

$$= 2\,\frac{(3 \times 1)\,(1)}{6.4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{16}.$$

Ex. 2. Evaluate $\int_{-\pi/2}^{\pi/2} \sin^5 x\,dx.$

By the theorem,

$$\int_{-\pi/2}^{\pi/2} \sin^5 x\,dx = \int_0^{\pi/2} \sin^5 x\,dx + \int_0^{\pi/2} \sin^5(-x)\,dx$$

$$= \int_0^{\pi/2} \sin^5 x\,dx - \int_0^{\pi/2} \sin^5 x\,dx = 0.$$

Theorem III :—

$$\boxed{\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx}$$

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx \quad \ldots \quad \ldots \quad \text{(i)}$$

Substituting $x = 2a - t$ in the second integral on the r. h. s. of (i), we get

$$\int_a^{2a} f(x)\,dx = -\int_a^{0} f(2a-t)\,dt$$

$$= \int_0^{a} f(2a-t)\,dt = \int_0^{a} f(2a-x)\,dx$$

Substituting the result in (i), we have

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx.$$

Ex. Evaluate (i) $\int_{-\pi}^{\pi} \sin^4 x \cos^3 x\,dx$, (ii) $\int_0^{2\pi} \sin^4 x \cos^3 x\,dx$.

(i) $\int_{-\pi}^{\pi} \sin^4 \cos^2 x \, dx = \int_0^{\pi} \sin^4 x \cos^2 x \, dx$

$+ \int_0^{\pi} \sin^4 (-x) \cos^2 (-x) \, dx$

[by the theorem II]

$= 2 \int_0^{\pi} \sin^4 x \cos^2 x \, dx$

$= 2 \left[\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx \right.$

$\left. + \int_0^{\pi/2} \sin^4 \left(\pi - x \right) \cos^2 (\pi - x) \, dx \right]$

[by theorem III]

$= 2 \left[2 \int_0^{\pi/2} \sin^4 x \cos^2 x \, dx \right]$

$= 4 \dfrac{3 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \dfrac{\pi}{2} = \dfrac{\pi}{8}$. [Ref. art. 14.7]

Similarly,

(i) $\int_0^{2\pi} \sin^4 x \cos^2 x \, dx = \int_0^{\pi} \sin^4 x \cos^2 x \, dx$

$+ \int_0^{\pi} \sin^4 (2\pi - x) \, dx$

[by Th. III].

$= 2 \int_0^{\pi} \sin^4 x \cos^2 x \, dx$

$= \dfrac{\pi}{8}$ [by above example]

Thus by the above Theorems, we have

$$\int_0^{\pi} \sin^n x\,dx = \int_0^{\pi/2} \sin^n x\,dx + \int_0^{\pi/2} \sin^n (\pi - x)\,dx$$

$$= \int_0^{\pi/2} \sin^n x\,dx + \int_0^{\pi/2} \sin^n x\,dz$$

$$= 2\int_0^{\pi/2} \sin^n x\,dx$$

Similarly,

$$\int_0^{\pi} \cos^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx + \int_0^{\pi/2} \cos^n (\pi - x)\,dx$$

$$= \int_0^{\pi/2} \cos^n x\,dx + (-1)^n \int_0^{\pi/2} \cos^n x\,dx$$

$$\left.\begin{aligned} &= 2\int_0^{\pi/2} \cos^n x\,dx \text{ if } n \text{ is even} \\ &= 0 \ldots \qquad \ldots \text{ if } n \text{ is odd} \end{aligned}\right\}$$

Thus :

$$\boxed{\begin{aligned} &\int_0^{\pi} \sin^n x\,dx = 2\,S_n \\ &\int_0^{\pi} \cos^n x\,dx = 2\,C_n, \text{ if } n \text{ is even} \\ &\qquad\qquad = 0, \text{ if } n \text{ is odd} \end{aligned}} \quad \ldots \quad (7)$$

14·6. Reduction formula for $\int \sin^m x \cos^n x\,dx$:—

Let $I_{(m,n)} = \int \sin^m x \cos^n x dx$, where m and n are both positive integers, then

$$I_{(m,n)} = \int \sin^m x \cos^n x \, dx$$

$$= \int \sin^{m-1} x \, (\sin x \cos^n x) \, dx$$

Integrating by parts, in which the bracketed term is the part that is integrated, and $\sin^{m-1} x$ is the part to be differentiated we have

$$I(m,n) = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^{n+2} x dx$$

Here the integrand

$$\sin^{m-2} x \cos^{n+2} x = \sin^{m-2} x \cos^n x \, [\, 1 - \sin^2 x \,]$$

$$= \sin^{m-2} x \cos^n x - \sin^m x \cos^n x$$

The expression for the integral $I(m,n)$ becomes

$$I(m, n) = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I(m-2,n) - \frac{m-1}{n+1} I(m,n)$$

Transposing the last term on the R. H. S. and multiplying by $\frac{n+1}{m+n}$, we have

$$I(m, n) = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I(m-2, n) \quad \ldots \quad (8)$$

Similarly in the $\int \sin^m x \cos^n xdx$, if we split up the term $\cos^n x$ and write integrand as $(\sin^m x \cos x) \cos^{n-1} x$ and integrate by parts, we get another reduction formula,

$$I(m, n) = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I(m, n-2) \quad \ldots \quad (9)$$

In (8) m is reduced by 2, n being kept the same and in (9) n is reduced by two.

14·7. Reduction Formula for $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$:–

... x dx, where m, n are positive

Then substituting limits of the integration 0 and $\frac{\pi}{2}$ in (8) and (9) above gives the reduction formulae as ...

$$I_{(m,n)} = \frac{m-1}{m+n} I_{(m-2,\, n)} \qquad \ldots \ldots \ldots \quad (10)$$

$$= \frac{n-1}{m+n} I_{(m,n-2)} \qquad \ldots \ldots \ldots \quad (11)$$

To obtain the value of the definite integral, we shall consider four different cases :—

(*i*) Firstly let m and n both be *even*, then by successive application of the formula (10),

$$I_{(m,\, n)} = \frac{m-1}{m+n} I_{(m-2,n)}$$

$$= \frac{m-1}{m+n} \frac{m-3}{m+n-2} I_{(m-4,\, n)}$$

$$= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \ldots \frac{1}{n+2} I_{(0,\, n)} \qquad \ldots \ldots \quad (12)$$

Now, $I_{(0,n)}$ is given by

$$I_{(0,n)} = \int_0^{\pi/2} \cos^n x \, dx$$

$$= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \ldots \frac{1}{2} \frac{\pi}{2} \text{ (as } n \text{ is even)}$$

Substituting for $I_{(0,\, n)}$ in (12) we get,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = I_{(m,\, n)}$$

$$= \left(\frac{m-1}{m+n} \frac{m-3}{m+n-2} \ldots \frac{1}{n+2} \right) \left(\frac{n-1}{n} \frac{n-3}{n-2} \ldots \frac{1}{2} \frac{\pi}{2} \right)$$

$$\text{or } I_{(m,n)} = \frac{[(m-1)\ (m-3)\ldots 3\cdot 1)]\ [(n-1)\ (n-3)\ldots 3.1]}{(m+n)\ (m+n-2)\ldots\ldots 4.2} \times \frac{\pi}{2} \quad (13)$$

(ii) Let m be *even* and n be *odd*, then from (12) we have

$$I_{(m,n)} = \frac{m-1}{m+n}\,\frac{m-3}{m+n-2}\,\cdot\cdot\,\frac{1}{n+2}\,I_{(0,\,n)}$$

$$\text{and } I_{(0,\,n)} = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\cdots\,\frac{2}{3}\,1 \text{ (as } n \text{ is odd)}$$

$$\therefore\ I_{(m,\,n)} = \frac{[\,(m-1)\ (m-3)\ 3.1\,]\,[\,(n-1)\ (n-3)\,..2\,]}{(m+n)\ (m+n-2)\ \ldots\ (n+2)\ n\ (n-2)\ \cdot\cdot\ 5\cdot 3} \times 1 \quad \ldots (14)$$

(iii) Let now n be *even* and m be *odd*, then

$$I_{(m,n)} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx$$

$$= \int_0^{\pi/2} \sin^m\left(\frac{\pi}{2} - x\right) \cos^n\left(\frac{\pi}{2} - x\right) dx,$$

[by Th. I of art. 14·5]

$$= \int_0^{\pi/2} \cos^m x \sin^n x\,dx = \int_0^{\pi/2} \sin^n x \cos^m x\,dx$$

$$= I_{(n,\,m)}$$

Thus as $I\ (m, n) = I\ (n, m)$ and n the index of sine term is even, and m the index of cosine term is odd, we can use the formula (14) and we have

$$I_{(m,\,n)} = I_{(n,\,m)}$$

$$= \frac{[\,(n-1)\ (n-3)\,..3\cdot 1\,]\,[\,(m-1)\ (m-3)\ \ldots\ 2\,]}{(m+n)\ (m+n-2)\ldots\ .5\cdot 3} \times 1.$$

Thus adjusting the numerator,

$$I_{(m,\,n)} =$$

$$\frac{[\,(m-1)\ (m-3)\ldots 4\cdot 2\,]\,[\,(n-1)\ (n-3)\ldots 3\cdot 1\,]}{(m+n)\ (m+n-2)\ \ldots\ .\ .\ 5\cdot 3} \times 1 \quad \ldots (15)$$

the same as the formula (14).

(iv) Lastly, let m and n both be *odd*.

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Here $I_{(m,n)} = \frac{m-1}{m+n} I_{(m-2,n)}$

Using this formula in succession, we shall have

$$I_{(m,n)} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdots \cdots I_{(3,n)}.$$

$$= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdots \frac{2}{n+3} I_{(1,n)}$$ using the same reduction formula.

$$\text{But } I_{(1,n)} = \int_0^{\pi/2} \sin x \cos^n x \, dx$$

$$= \left[-\frac{\cos^{n+1} x}{n+1} \right]_0^{\pi/2} = \frac{1}{n+1}.$$

Substituting this value above we have

$$I_{(m,n)} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdots \frac{2}{n+3} \times \left(\frac{1}{n+1}\right).$$

To adjust this in a convenient form, we multiply this expression in numerator and denominator by

$(n-1)(n-3) \ldots\ldots 4 \cdot 2$ and so we have finally

$$I_{(m,n)} = \frac{[(m-1)(m-3)\ldots 2][(n-1)(n-3)\ldots 4\cdot 2]}{(m+n)(m+n-2)\ldots(n+3)(n+1)(n-1)(n-3)\ldots 4\cdot 2} \quad \ldots(16)$$

All the above four formulae can be summed up into,

$$\boxed{I_{(m,n)} = \frac{[(m-1)(m-3)\ldots 2 \text{ or } 1][(n-1)(n-3)\ldots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots\ldots 2 \text{ or } 1} \times k} \quad (17)$$

where $k = \pi/_2$, if m and n are both *even*.

$= 1$, for all other values of m, n.

Note the way in which the numerator and the denominator are written.

14·8. Value of $\int_0^{\pi} \sin^m x \cos^n x \, dx$:—

Now by Theorem II of art. 14·5, we have

$$\int_0^{\pi} \sin^m x \cos^n x \, dx = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$+ \int_0^{\pi/2} \sin^m (\pi - x) \cos^n (\pi - x) \, dx$$

$$= \int_0^{\pi/2} \sin^m x \cos^n x \, dx + (-1)^n \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= 2 \int_0^{\pi/2} \sin^m x \cos^n x \, dx, \text{ if } n \text{ is } even.$$

$$= 0 \quad .. \quad .. \quad .., \text{ if } n \text{ is } odd.$$

Illustrative Examples :—

Let us now see the use of the above formulae in evaluating the integrals

Example 1. Evaluate (i) $\int_0^{\pi/2} \sin^3 \theta \cos^4 \theta \, d\theta$; (ii) $\int_0^{\pi/4} \sin^7 2\theta \, d\theta$

(ii) $\int_0^{\pi} (1 - \cos \theta)^3 \, d\theta$; (iv) $\int_0^{2} x^3 \sqrt{(2-x)} \, dx$.

(i) $$\int_0^{\pi/2} \sin^3 \theta \cos^4 \theta \, d\theta = \frac{(2) \cdot (3 \,.\, 1)}{(3+4)(3+4-2)(3+4-4)(3+4-6)}$$

Using formula .. (17)

$$= \frac{2 \cdot 3}{7.5 \cdot 3.1} = \frac{2}{35}.$$

(ii) $\int_0^{\pi/4} \sin^7 2\theta \, d\theta$. Put $2\theta = x$, the integral becomes

$$\int_0^{\pi/4} \sin^7 2\theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^7 x \, dx$$

$$= \frac{1}{2} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}, \text{ by formula (3)}$$

$$= \frac{8}{35}.$$

(iii) $$\int_0^{\pi} (1 - \cos\theta)^3 \, d\theta = \int_0^{\pi} \left[2 \sin^2 \frac{\theta}{2}\right]^3 d\theta. = 8 \int_0^{\pi} \sin^6 \frac{\theta}{2} \, d\theta.$$

Putting $x = \theta/2$

$$= 8 \cdot 2 \int_0^{\pi/2} \sin^6 x \, dx, \quad \text{by formula (3)}$$

$$= 16. \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 5 \frac{\pi}{2}.$$

(iv) $$\int_0^2 x^3 \sqrt{2 - x} \, dx.$$

Put $x = 2 \sin^2\theta$, $dx = 4 \sin\theta \cos\theta \, d\theta$; when $x = 0$, $\theta = 0$; and $x = 2$ $\sin\theta = 1$, $\theta = \pi/2$. Therefore the integral is

$$\int_0^{\pi/2} (2)^3 \sin^6 \theta \, (2)^{1/2} \cos\theta. \; 4 \sin\theta \cos\theta \, d\theta$$

$$= 32 \sqrt{2} \int_0^{\pi/2} \sin^7\theta \cos^2\theta \, d\theta$$

$$= 32 \sqrt{2} \, \frac{[6 \cdot 4.2]\,[1]}{9.7.5.3.1} \quad \text{by formula (17)}$$

$$= 32 \sqrt{2} \, \frac{16}{315} = \frac{512 \sqrt{2}}{315}.$$

Example 2. By considering the value of $\int_0^1 (1 - x^2)^n \, dx$, show that

$$1 - \frac{n}{1.3} + \frac{n(n-1)}{1.2.5} - \frac{n(n-1)(n-2)}{1.2.3.7} + \ldots$$
$$= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1}$$

Let $I = \int_0^1 (1 - x^2)^n \, dx$.

Put $x = \sin\theta$, $dx = \cos\theta \, d\theta$

$$\therefore \quad I = \int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \ldots \ldots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

$$= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \ldots \ldots \frac{2n}{2n+1} \qquad \ldots \text{(i)}$$

Next expanding $(1 - x^2)^n$ by the binomial theorem, we have,

$$I = \int_0^1 (1 - x^2)^n \, dx$$

$$= \int_0^1 \left[1 - nx^2 + \frac{n.(n-1)}{2!} x^4 - \frac{n.(n-1)(n-2)}{3!} x^6 + \ldots \right] dx$$

$$= 1 - n\frac{1}{3} + \frac{n(n-1)}{1.2} \cdot \frac{1}{5} - \frac{n(n-1)(n-2)}{1.2.3} \frac{1}{7} + \ldots \qquad \ldots \text{(ii)}$$

From (i) and (ii) we have;

$$1 - \frac{n}{1.3} + \frac{n(n-1)}{1.2.5} - \frac{n(n-1)(n-2)}{1.2.3 \cdot 7} + \ldots = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \ldots \ldots \frac{2n}{2n+1}.$$

Example 3. If $U_n = \int_0^{\pi/4} \tan^n \theta \, d\theta$, show that $U_n = \frac{1}{n-1} - U_{n-2}$

hence calculate $\int_0^{\pi/4} \tan^6 x \, dx$

$$U_n = \int_0^{\pi/4} \tan^n \theta \, d\theta = \int_0^{\pi/4} \tan^{n-2}\theta \, (\sec^2\theta - 1) \, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2}\theta \sec^2\theta \, d\theta - U_{n-2}$$

In the first integral, if we put $x = \tan\theta$, it becomes

$$\int_0^1 x^{n-2}\,dx = \frac{1}{n-1}$$

We have, $U_n = \dfrac{1}{n-1} - U_{n-2}$.

Next $\int_0^{\pi/4} \tan^6 x\,dx$ by the successive use of this formula is

$$U_6 = \frac{1}{5} - U_4$$

$$= \frac{1}{5} - \left[\frac{1}{3} - U_2\right]$$

$$= \frac{1}{5} - \left[\frac{1}{3} - \left(1 - U_0\right)\right]$$

and $U_0 = \int_0^{\pi/4} \tan^0 x\,dx = \int_0^{\pi/4} dx = \dfrac{\pi}{4}$

$$\therefore \int_0^{\pi/4} \tan^6 x\,dx = U_6$$

$$= \frac{1}{5} - \left[\frac{1}{3} - \left(1 - \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$$

$$= \frac{13}{15} - \frac{\pi}{4}$$

14.9. Beta and Gamma Functions :–

We shall consider here briefly the two important functions, known as Beta and Gamma Functions, which facilitate the evaluation of certain definite integrals.

We define $\int_0^{\infty} e^{-x}\, x^{n-1}\, dx$, which is a function of n $(n > 0)$ as a Gamma Function and denote it by $\Gamma(n)$. Thus

$$\Gamma(n) = \int_0^{\infty} e^{-x}\, x^{n-1}\, dx \quad \ldots \quad (18)$$

Let us prove some simple properties of this function.

We have $\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx.$

Integrating by parts

$$\Gamma(n+1) = \left[-x^n e^{-x}\right]_0^\infty + n \int_0^\infty e^{-x} . x^{n-1} \, dx = n\,\Gamma(n)$$

Thus $\quad \Gamma(n+1) = n\,\Gamma(n) \quad \ldots \quad \ldots \quad \ldots \quad (19)$

which is the reduction formula for the Gamma function.

If n is a positive integer, then by successive application of formula (19), we get

$$\Gamma(n+1) = n(n-1)(n-2) \ldots\ldots 2\cdot1\,\Gamma(1)$$

and as $\Gamma(1) = \int_0^\infty e^{-x} \, dx = \left[-e^{-x}\right]_0^\infty = 1$ and so

$$\Gamma(n+1) = n! \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

If we substitute $x = z^2$ in (18), we get

$$\Gamma(n) = 2\int_0^\infty e^{-z^2} z^{2n-1} \, dz \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

We next define $\int_0^1 x^{m-1} (1-x)^{n-1} \, dx$, which is a function of m, n, as a Beta Function and denote it by B (m,n).

Thus

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \quad \ldots \quad \ldots \quad (22)$$

Then by $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$, we have

$$\int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \int_0^1 x^{n-1} (1-x)^{m-1} \, dx$$

and so $\quad B(m,n) = B(n,m) \quad \ldots \quad \ldots \quad \ldots \quad (23)$

The important relation between Beta and Gamma functions is

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \dots \quad \dots \quad \dots \quad (24)$$

The proof of the relation (24) is deferred to a latter chapter as it involves double integration [Ref. Chap. XVI, art 16·4]

If in (22) we substitute $x = \sin^2\theta$, we get

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

In this relation writing $p = 2m - 1$ and $q = 2n - 1$,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \tfrac{1}{2} B(m, n) = \tfrac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q}{2} + 1\right)} \qquad \dots \quad \dots \quad (25)$$

a relation which facilitates the evaluation of the definite integral.

In (22), if we put $x = \dfrac{y}{1+y}$, then we get

$$B(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} \, dy \qquad \dots \quad \dots \quad (26)$$

In (25) taking $p = q = 0$, we have

$$\int_0^{\pi/2} d\theta = \frac{[\Gamma(\frac{1}{2})]^2}{2\,\Gamma(1)} = \tfrac{1}{2}[\Gamma(\tfrac{1}{2})]^2 \text{ and as } \Gamma(1) = 1, \qquad \dots \quad (27)$$

$\therefore$ $[\Gamma(\frac{1}{2})]^2 = \pi$ and so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Example 1. $\int_0^{\pi/2} \sin^5\theta \cos^3\theta \, d\theta.$

Using relation (25),

$$\int_0^{\pi/2} \sin^5\theta \cos^3\theta \, d\theta = \frac{\Gamma(3)\,\Gamma(2)}{2\,\Gamma(5)} = \frac{2!\;1!}{2 \times 4!}$$

$$= \frac{2.1}{2\cdot 1.2\cdot 3.4} = \frac{1}{24}$$

Example 2. Evaluate $\int_0^{\infty} x^2 e^{-h^2 x^2}\, dx.$

Put $z = h^2 x^2$, and the integral is

$$I = \frac{1}{2h^3} \int_0^{\infty} z^{\frac{1}{2}} e^{-z}\, dz$$

$$= \frac{1}{2h}\, \Gamma\left(\frac{3}{2}\right) = \frac{1}{2h^3} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4h^3}$$

Example 3. Prove that $y\, B(x+1, y) = x\, B(x, y+1)$

$$y\, B(x+1, y) = y.\, \frac{\Gamma(x+1)\,\Gamma(y)}{\Gamma(x+1+y)}$$

$$= y.\, \frac{x\,\Gamma(x)\,\Gamma(y)}{\Gamma(x+1+y)}$$

$$= x.\, \frac{\Gamma(x).\,\Gamma(y+1)}{\Gamma(x+1+y)}$$

$$= x\, B(x, y+1).$$

Example 4. Evaluate the integrals

(i) $\int_0^{\infty} x^{n-1} e^{-ax} \cos bx\, dx$ (ii) $\int_0^{\infty} x^{n-1} e^{-bx} \sin bx\, dx$

when n is an integer.

Consider the integral $\int_0^{\infty} e^{-(a-ib)x} x^{n-1}\, dx$

Put $(a - ib)\, x = t$ $\therefore$ $(a - ib)\, dx = dt$

$$\therefore \int_0^{\infty} e^{-(a-ib)x} x^{n-1}\, dx = \int_0^{\infty} e^{-t} \left(\frac{t}{a - ib}\right)^{n-1} \frac{dt}{a - ib}$$

$$= \frac{1}{(a-ib)^n}\int_0^\infty e^{-t}\, t^{n-1}\, dt = \frac{\Gamma(n)}{(a-ib)^n} \qquad \text{(i)}$$

Now let

$$a - ib = re^{-i\theta} \quad \therefore\ r^2 = a^2 + b^2,\ \theta = \tan^{-1}\frac{b}{a}$$

$$\therefore\ (a-ib)^n = r^n e^{in\theta}.$$

$\therefore$ from (i), we have

$$\int_0^\infty e^{-(a+ib)x}\, x^{n-1}\, dx = \frac{e^{in\theta}}{r^n}\,\Gamma(n)$$

Equating real and imaginary parts, we get

$$\int_0^\infty x^{n-1}\, e^{-ax} \cos bx\, dx = \frac{\Gamma(n)}{r^n}\cos n\theta$$

$$\int_0^\infty x^{n-1}\, e^{-bx} \sin bx\, dx = \frac{\Gamma(n)}{r^n}\sin n\theta.$$

Examples : XIV

1. Prove that :—

(i) $\displaystyle\int_0^{\pi/2} \sin^2\theta \cos^6\theta\, d\theta = \frac{5}{256}\pi.$ (ii) $\displaystyle\int_0^{\pi} \sin^2\theta\,(1+\cos\theta)^4\, d\theta = \frac{21}{16}\pi.$

(iii) $\displaystyle\int_4^{6} \sin^4\pi x \cos^2 2\pi x = \frac{7}{16}$ (iv) $\displaystyle\int_0^{\pi} x \sin^5 x \cos^4 x\, dx = \frac{8\pi}{315}$

(v) $\displaystyle\int_0^{\pi} x\cdot\cos^6 x\, dx = \frac{5\pi^2}{32}$ (iv) $\displaystyle\int_0^{1} x^6\sqrt{1-x^2}\, dx = \frac{5}{256}\pi.$

(vii) $\displaystyle\int_0^{1/2} x^3\sqrt{1-4x^2}\, dx = \frac{1}{120}.$ (viii) $\displaystyle\int_0^{2a} x\sqrt{2ax-x^2}\, dx = \frac{\pi a^3}{2}$

(ix) $\displaystyle\int_0^{\infty} \frac{x^2}{(1+x^2)^{7/2}}\, dx = \frac{8}{45}$

2. Prove :—

(i) $\displaystyle\int_0^{1} \frac{x^7}{\sqrt{1-x^4}}\, dx. = \frac{1}{3}.$

(ii) $\displaystyle\int_0^1 (1 - x^{1/n})^m \, dx = \frac{m!\, n!}{(m+n)!}$

(iii) $\displaystyle\int_0^1 \frac{x^{2n}}{\sqrt{1 - x^2}} \, dx = \frac{(2n)!}{2^{2n}\,(n!)^2} \cdot \frac{\pi}{2}.$

(iv) $\displaystyle\int_0^\infty \frac{dx}{(x^2 + 1)^n} = \frac{(2n-2)!}{2^{2n-2}\,[(n-1)!]^2} \, \frac{\pi}{2}$

(v) $\displaystyle\int_0^{2a} x^m (2ax - x^2)^{\frac{1}{2}} \, dx = \frac{a^{m+2}\,\pi\,(2m+1)!}{2^m\, m!\,(m+2)!}$

3. If $f(m, n) = \int x^m (1 - x)^n \, dx$, show that

$$f(m, n) = \frac{x^{m+1}(1 - x)^n}{m + n + 1} + \frac{n}{m + n + 1} f(m, n - 1).$$

Hence show that $\displaystyle\int_0^1 x^m (1 - x)^n \, dx = \frac{m!\, n!}{(m + n + 1)!}$

4. If $u_n = \displaystyle\int_0^{\pi/2} \theta \cos^n \theta \, d\theta$, prove that

$$u_n = -\frac{1}{n^2} + \frac{n-1}{n} u_{n-2}.$$ Evaluate u_4.

[Hint : Write the integrand as $\cos^{n-1} \theta\, (\theta \cos \theta)$ and integrate by parts]

[Ans. $\dfrac{3\pi^2}{64} - \dfrac{1}{4}$.]

5. If $I_n = \displaystyle\int_0^{\pi/2} x^n \sin x \, dx$, $(n > 1)$, prove that

$$I_n + n(n-1)\, I_{n-2} = n\, (\pi/2)^{n-1}$$

6. If $I_n = \displaystyle\int_{\pi/4}^{\pi/2} \cot^n \theta \, d\theta$, $(n > 2)$, prove that

$$I_n = \frac{1}{n-1} - I_{n-2}$$

and hence evaluate $\int_{\pi/4}^{\pi/2} \cot^6 \theta .\ d\theta$ $\left[\text{Ans. } \frac{13}{15} - \frac{\pi}{4}\right]$

7. By using the substitution $1 + x = 2\cos^2 \theta$, evaluate the integral $\int_{-1}^{1} (1 + x)^m (1 - x)^n \, dx$, where m, n are positive integers.

8. Obtain a reduction formula for $\int \sec^n \theta \, d\theta$, by writing

$$I_{n-2} = \int \sec^{n-2} \theta \, d\theta = \int \frac{d\theta}{\cos^{n-2}\theta} = \int \frac{\cos\theta \, d\theta}{\cos^{n-1}\theta} = \int \frac{d(\sin\theta)}{\cos^{n-1}\theta}$$

integrating by parts and rearranging suitably. Use the formula to find $\int_0^{\pi/4} \sec^5\theta \, d\theta$.

9. If $In = \int_0^{\infty} e^{-x} \sin^n x \, dx$, $(n > 2)$, then prove that

$$(1 + n^2) I_n = n(n - 1) I_{n-2}$$ and hence evaluate I_4 [Ans. 24/85]

10. If $In = \int_0^{\pi/2} x^n \cos ax \, dx$, show that

$$I_n = \frac{1}{a} (\pi/2)^n [\sin(a\pi/2) + \frac{2n}{a\pi} \cos(a\pi/2)] - \frac{n(n-1)}{a^2} I_{n-2}$$

Hence evaluate $\int_0^{\pi/2} x^3 \cos x . \, dx$.

11. If $Im, n = \int \cos^m x \sin nx dx$, prove that

$$(m + n) I_{m,n} = -\cos^m x \cos nx + m I_{m-1, n-1}$$

and if $I_{m,n} = \int \cos^m x \cos nx \, dx$, then

$$I_{m,n} = \frac{\cos^m x \sin nx}{m + n} + \frac{m}{m + n} I_{m-1, n-1},$$

hence prove that

$$\int_0^{\pi/2} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}.$$

12. Prove that

$$\int_0^{\pi/2} \sin^{2m-1} x . \cos^{2n-1} x \, dx = \frac{1.2\cdot3 \ldots (m-1)}{n\,(n+1)\,(n+2) \ldots (n+m-1)} \cdot \frac{1}{2}$$

13. Evalute $\int_0^{\pi/2} \sin^{2m} \theta \, d\theta$, where m is a positive integer.

The expression $\dfrac{1-a}{(1-a\sin^2\theta)^{3/2}} - (1-a\sin^2\theta)^{1/2}$, where $0 < a < 1$ is expanded in ascending powers of a and the coefficient of a^n is denoted by u_n.

$$\text{rove that} \int_0^{\pi/2} u_n \, d\theta = 0.$$

14. Show that

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{\sin^2 \dfrac{\alpha}{2} - \sin^2 \dfrac{\theta}{2}}} = \pi \left[1 + \left(\frac{1}{2}\right)^2 a^2 + \left(\frac{1}{2} \cdot \frac{3}{4}\right)^2 a^4 + \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}\right)^2 a^6 + \ldots \right]$$

where $a = \sin \dfrac{\alpha}{2}$.

15. Show that $\displaystyle\int_0^{\infty} \frac{d\theta}{\cosh^n \theta} = \frac{n-2}{n-1} \int_0^{\infty} \frac{d\theta}{\cosh^{n-2}\theta}$

16. Prove that if $I(m, n) = \displaystyle\int \frac{\sin^m\theta}{\cos^n \theta} d\theta$, then

$$I(m, -n) = \frac{\sin^{m-1}\theta}{(n-1)\cos^{n-1}\theta} - \frac{m-1}{n-1} I[m-2, -(n-2)].$$

[Hint Write the integral as $-\displaystyle\int \frac{\sin^{m-1}\theta}{\cos^n \theta} d(\cos\theta)$]

17. If $U_n = \int x^n e^x \, dx$, prove that $U_n = x^n e^x - n\,U_{n-1}$

Hence evaluate U[illegible]

18. If $U_n = \int_0^{\pi/4} \sin^{2n} x \, dx$, prove that

$$U_n = \left(1 - \frac{1}{2n}\right) U_{n-1} - \frac{1}{n.2^{n+1}}.$$

19. If $I_n = \int x^n \sin ax \, dx$, show that

$$a^2 I_n = -ax^n \cos ax + nx^{n-1} \sin ax - n(n-1) I_{n-2},$$

Evaluate $\int_0^{\pi/4} x^4 \sin 2x \, dx$. $\left[\text{Ans.} \quad \frac{\pi^3}{64} - \frac{3\pi}{8} + \frac{3}{4}\right]$.

20. If $I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta$, prove that, when n is positive integer

$n(I_{n-1} + I_{n+1}) = 1$. Hence or otherwise prove that

$$\int_0^a x^5 (2a^2 - x^2)^{-3} dx = \tfrac{1}{2}(\log 2 - \tfrac{1}{2}).$$

21. If $I_{m,n} = \int x^m (\log x)^n dx$, then prove that

$$I_{m,n} = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{n+1} I_{m,n-1}$$

22. If $I_n = \int x^n (a-x)^{1/2} dx$, prove that

$$(2n+3) I_n = 2an \, I_{n-1} - 2x^n (a-x)^{3/2}.$$

Evaluate $\int_0^a x^2 (ax - x^2)^{1/2} dx$. $\left[\text{Ans} \quad \frac{5\pi a^4}{128}\right]$

23. If $I_n = \int \frac{x^n}{(a^2+x^2)^{3/2}} dx$, show that

$$(n-2) I_n = \frac{x^{n-1}}{(a^2+x^2)^{1/2}} - (n-1) a^2 I_{n-2}$$

and hence evaluate $\int_0^1 \frac{x^5}{(3+x^2)^{3/2}} dx$. [Ans. 0.023]

24. If $I_n = \int x^n (a^2 - x^2)^{1/2} dx$, show that

$$(n+2) I_n = -x^{n-1} (a^2 - x^2)^{3/2} + a^2 (n-1) I_{n-2}$$

25. Show that if

(i) $I_n = \int (x^2 + a^2)^{n/2}\, dx$, $I_n = \dfrac{x(x^2 + a^2)^{n/2}}{n+1} + \dfrac{na^2}{n+1}\int (x^2+a^2)^{\frac{n}{2}-1} dx$

(ii) $I_n = \int \dfrac{dx}{(x^2+a^2)^n}$, $I_n = \dfrac{x}{(2n-2)\, a^2 (x^2 + a^2)^{n-1}} + \dfrac{2n-3}{(2n-2)a^2} I_{n-1}$

(iii) $I_n = \int x^n \sqrt{2ax - x^2}\, dx$, $I_n = -\dfrac{x^{n-1}(2ax-x^2)^{3/2}}{n+2} + \dfrac{a(2n+1)}{n+2} I_{n-1}$

(iv) $I_n = \int_0^a (a^2 - x^2)^n\, dx$, $I_n = \dfrac{2na^2}{2n+1} I_{n-1}$

26. Show that

$$\int \cos^{2n}\phi\, d\phi = \frac{1}{2n}\tan\phi \cos^{2n}\phi + \left(1 - \frac{1}{2n}\right)\int \cos^{2n-2}\phi\, d\phi.$$

27. By means of $\int x^{2n+1}(1 - x^2)^{-1/2}\, dx$, show that

$$\frac{1}{2n+2} + \frac{1}{2}\,\frac{1}{2n+4} + \frac{1}{2}\,\frac{3}{4}\,\frac{1}{2n+6} + \;\ldots\;\ldots \text{ ad inf.} = \frac{2.4.6\ldots\ldots 2n}{3.5.7\ldots\ldots(2n+1)}.$$

28. Prove that

$$\int_0^1 x^{4m+1}\sqrt{\frac{1-x^2}{1+x^2}}\, dx = \frac{1.3.5\ldots\ldots(2m-1)}{2.4.6\ldots\ldots 2m}\,\frac{\pi}{4} - \frac{2.4.6.\ldots\ldots(2m)}{3.5.7\ldots\ldots(2m+1)}\,\frac{1}{2}.$$

29. Prove that

(i) $\displaystyle\int_0^1 x^m (1 - x^n)^p\, dx = \frac{\Gamma(p+1)\Gamma\left(\frac{m+1}{n}\right)}{n\Gamma\left(p+1+\frac{m+1}{n}\right)}$

(ii) $1.3.5\ldots\ldots(2n-1) = \dfrac{2^n\,\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}$.

(iii) $B(m, n) = B(m, n+1) + B(m+1, n)$

(iv) $B(x+1, y) = \dfrac{x}{x+y} B(x, y)$

(v) $\Gamma(m)\,\Gamma(m+\frac{1}{2})\quad \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m).$

(vi) $B(m, m) = 2^{1-2m}\,B(m, \frac{1}{2})$

(vii) $$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = B(m, n)$$

(viii) $$\int_1^\infty \frac{x^{\frac{n}{2}-1}}{(1+x)^n}\,dx = \frac{1}{2}\,B\left(\frac{n}{2}, \frac{n}{2}\right)$$

(ix) $$\left(\int_0^{\pi/2} \sin^p x\,dx\right)\left(\int_0^{\pi/2} \sin^{p+1} x\,dx\right) = \frac{\pi}{2(p+1)}$$

(x) $$\int_1^\infty \frac{dx}{x^{p+1}(x-1)^q} = B(p+q, 1-q), \text{ if } -p < q < 1$$

(xi) $$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{B(m, n)}{a^n(a+b)^m}$$

(xii) $$\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = 2\int_0^{\pi/2} \sqrt{\tan\theta}\;d\theta = \pi\sqrt{2}.$$

(xiii) $$\int_a^b (x-a)^m\,(b-x)^n\,dx = (b-a)^{m+n+1}\,B(m+1, n+1)$$

(xiv) $$B(m, n)\,B(m+n, p) = \frac{\Gamma(m)\,\Gamma(n)\,\Gamma(p)}{\Gamma(m+n+p)}$$

(xv) $$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m}\,B(n, m)$$

(xvi) $$\int_0^{\pi/2} \frac{\sin^{2m-1} x.\cos^{2n-1} x}{(a\sin^2 x + b\cos^2 x)^{m+n}}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{2a^m b^n\,\Gamma(m+n)}$$

(xvii) $$\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}$$

30. **Prove that**

(i) $\displaystyle\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$, if $n > 0$

and hence evaluate $\displaystyle\int_0^\infty \operatorname{sech}^8 x \, dx \left[\text{ use } y = e^{2x}, \text{ ans. } \frac{16}{35}\right]$

31. $$\int_0^\pi \frac{\sin^{n-1} x}{(a+b\cos x)^n} dx = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} B\left(\frac{n}{2}, \frac{n}{2}\right)$$

32. If $B(n, 3) = \frac{1}{3}$ and n is a positive integer, find n. [Ans. $n = 1$]

33. Prove that

(i) $\displaystyle\int_{-\pi/4}^{\pi/4} (\cos\theta + \sin\theta)^{1/3} d\theta = 2^{1/6}\sqrt{\pi}\,\frac{\Gamma(2/3)}{\Gamma(7/6)}$

(ii) $\displaystyle\int_3^7 \sqrt[4]{(7-x)(x-3)}\, dx = \frac{2\,[\Gamma(1/4)]^2}{3\sqrt{\pi}}$

34. Evaluate

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^m}}$ (ii) $\displaystyle\int_0^1 x^{a-1}\left(\log\frac{1}{x}\right)^{n-1} dx.$

(iii) $\displaystyle\int_0^1 x^3(1-\sqrt{x})^5\, dx.$ (iv) $\displaystyle\int_0^\infty \frac{x^a}{a^x}\, dx\ (a > 1)$

(v) $\displaystyle\int_0^n x^n(n-x)^p\, dx.$ (vi) $\displaystyle\int_0^a (a^6-x^6)^{1/6}\, dx.$

(vii) $\displaystyle\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$ (viii) $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$

(ix) $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$ (x) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ (xi) $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^3}\, dx$

(xii) $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$ (xiii) $\int_0^1 (x \log x)^3\, dx$ (xiv) $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

$\left[\right.$ Ans. (i) $\frac{\sqrt{\pi}}{m} \cdot \frac{\Gamma\left(\frac{1}{m}\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}$. (ii) $\frac{\Gamma(n)}{a^n}$ (iii) $\frac{1}{5148}$

(iv) $\frac{\Gamma(a+1)}{(\log a)^{a+1}}$. (v) n^{n+p+1} B $(n+1, p+1)$] (vi) $\frac{a^2\{\Gamma(\frac{1}{3})\}^2}{2\,\Gamma(\frac{2}{3})}$

(vii) 1/5005 (viii) 0 (ix) $\frac{\pi\sqrt{2}}{4}$

(x) $\{\Gamma(1/4)\}^2/2\sqrt{\pi}$ (xi) $\frac{\pi}{32}$ (xii) $\frac{3\sqrt{\pi}}{2}$

(xiii) $-3/_{128}$ (xiv) $\sqrt{\pi}$ $\left.\right]$

35. Show that

(i) $\int_0^{\pi/2} (\tan x)^n\, dx = \frac{\pi}{2} \sec \frac{n\pi}{2}$ where $-1 < n < 1$.

(ii) $\int_0^1 \frac{x^2}{(1-x^4)^{1/2}}\, dx \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$.

(iii) $\int_0^\infty x\, e^{-ax} \cos bx\, dx = \frac{a^2-b^2}{(a^2+b^2)^2}$

CHAPTER I

INTRODUCTION TO DIFFERENTIAL EQUATIONS

1·1. Definitions :— An equation involving differential coefficients or differentials is called a differential equation. If the differential coefficients be ordinary, the equation is called ordinary differential equation, and if they be partial differential coefficients, the equation is called partial differntial equation.

Thus

$$\frac{dy}{dx} - \cos x = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(a)}$$

$$y = x\frac{dy}{dx} + \frac{c}{\left(\frac{dy}{dx}\right)} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(b)}$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(c)}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(d)}$$

$$\left.\begin{aligned} \frac{dx}{dt} + 4x + 3y &= t \\ \frac{dy}{dt} + 2x + 5y &= e^t \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(e)}$$

are all differenial equations, out of which (d)is a partial differential equation, and the others are ordinary differential equations.

The *order* of a differential equation is the order of the hightest derivative occurring in the differential equation. Thus (c) is a differential equation of the second order. The *degree* of a differential equation is the degree of the highest derivative, when the differential coefficients are rational and free from fractions. Thus equation (c) above is the first degree equation, (b) written so as to remove the fraction becomes

$$y\left(\frac{dy}{dx}\right) = x\left(\frac{dy}{dx}\right)^2 + c$$

and so is of the second degree.

1·2. Formation of a differential equation :— Mathematically, a differential equation is formed in an attempt to eliminate arbitrary constant in the relation of the variables. Thus to eliminate the two constants A, B from the relation

$$y = Ae^{3x} + Be^{2x} \quad \ldots \quad (1)$$

we need two more equations, which can be easily formed through successive differentiation, that is

$$\left.\begin{aligned} \frac{dy}{dx} &= 3Ae^{3x} + 2Be^{2x} \\ \frac{d^2y}{dx^2} &= 9Ae^{3x} + 4Be^{2x} \end{aligned}\right\} \quad \ldots \quad (2)$$

Multiplying the first equation of (2) by -5, and (1) by 6, and adding all of them eliminates A, B, leading to the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \quad \ldots \quad (3)$$

The relation (1) is called the solution of the differential equation (3). Here we note, that to eliminate two arbitrary constants, we require two more equations in addition to the given relation, leading us to second order derivatives, and so a differential equation of the second order, and it is then clear that elimination of three arbitrary constants will lead us to a differential equation of the third order. Thus elimination of n arbitrary constants will bring us to a differential equation of the nth order; and conversely the solution of a differential equation of the nth order will in general contain n arbitrary constants.

1·3. Types of Solution :— Any relation between the dependent and independent variables not involving the derivatives, which satisfies a differential equation is called a solution of the differential equation.

The solution which involves as many arbitrary constants as the order of the differential equation is called the *general*

solution. Thus from (1.2) above, we see that

$$y = Ae^{3x} + Be^{2x}$$

is the general solution of the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

as the solution contains two arbitrary constants and the differential equation is of the second order. Similarly,

$$y = \sin x + c.$$

is a general solution of the equation.

$$\frac{dy}{dx} - \cos x = 0$$

as can be verified by actual substitution.

The solution in which arbitrary constants in the G. S. (i. e. general solution) are given specific numerical values is called a *particular* solution. Thus in the above if we take A = 2 and B = 7, we have

$$y = 2e^{3x} + 7e^{2x}$$

which satisfies the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

and so is a particular solution of the equation. The particular values taken by the arbitrary constants, depend upon what are known as initial or boundary conditions of the problem. This point will be made more clear, when we shall study applied differential equations.

There is one more type of solution, which is called *a singular* solution. The singular solution is a relation between the variables, which does not involve arbitrary constants, and still satisfies the differential equation. A singular solution cannot be obtained (as the particular solution can be obtained) from the general solution by giving any particular numerical values to the arbitrary constants.

To make this clear, let us consider the equation

$$y = x\frac{dy}{dx} + \frac{1}{\left(\frac{dy}{dx}\right)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

It can be verified very easily that its general solution is

$$y = cx + \frac{1}{c}$$

where c is an arbitrary constant.

Another solution of this equation is $y^2 = 4x$, which can also be verified by substitution. This is the sigular solution of the equation; and as can be seen, it cannot be obtained from the general solution by giving any particular value to c, the arbitrary constant.

1·4. Engineer's Approach to Differential equations :— Differenial Equations play an impotant role in the study of Engineering Problems, and the approach of an Engineering student to the differential equation differs from that of a student of Mathematics. For the engineering student, the study of a differential equation can be divided in three stages.

(a) Formation of differential equation, from the given Physical or Engineering problem.

(b) To obtain the solution of the differential equation, and fixing the values of the arbitrary constants, with the help of given conditions.

(c) Physical interpretation of the mathematical solution.

For instance, suppose we are dealing with an electrical circuit, containing a resistance R, an inductance L, and an e. m. f. E all in series.

Let i be the current flowing in the circuit at time t. We need to know the way in which the current varies with time, and how it is affected by the values of the resistance and the inducance. This is the Physical Engineering Problem.

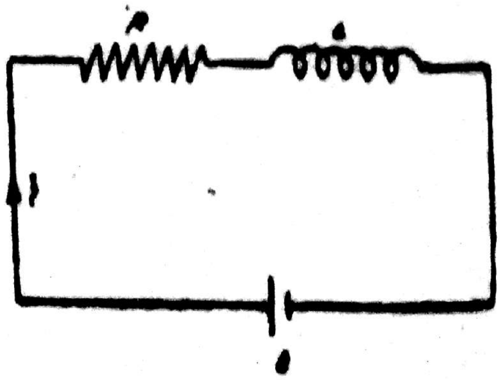

Fig. 1

In the first stage, we transform this into a differential equation. Using Kirchhoff's Laws for circuit, since the potential

drops across R and L are Ri and $L\frac{di}{dt}$ respectively, we have

$$L\frac{di}{dt} + Ri = E.$$

This is the mathematical statement of the electrical circuit. The next stage is to solve this differential equation. By the methods which follow, we obtain the solution of this differential equation as :

$$i = ce^{-\frac{Rt}{L}} + \frac{E}{R}.$$

Initially if $i = 0$ when $t = 0$, then substituting these in the above relation we get $c = -E/R$.

$\therefore$ The required solution is

$$i = \frac{E}{R}\left\{1 - e^{-\frac{Rt}{L}}\right\}.$$

In the final stage, we try to interpret this solution in terms of the physical nature of the problem. We see here that as t increases, the current i goes on increasing, taking ultimately the maximum value E/R. This relation also help us to note how i is affected by the value of the resistance R and the inductance L in the circuit.

The mathematics student is only interested in the solution of a differential equation and in the analytical theory of the solution of the differential equation. The Engineering student's approach is more practical.

In the rest of this section, we shall first see different method to solve the differential equations and then the application of the differential equations to the Engineering and Physical Problems.

Examples

Translate the following statements into mathematical symbols :

1. The slope of a certain plane curve is everywhere equal to the difference of ordinate and abscissa.

2. The acceleration of a rocket of mass m directed straight upward is retarded by both a constant gravity force and a force proportional to its speed.

3. An electrical circuit contains a resistance R, an inductance L, a capacity C, and an e. m. F. $E \sin \omega t$. all in series.

Give order and degree of each of the following differ ntials equations :

4. $y''^2 + y'^4 = c^4$.

5. $y''' + y'^2 = 3$.

6. $y'' + x/y'' = 1$.

7. $y' + x = (y - xy')^{-2}$

8. $y'' = \sqrt{1 + y'^2}$.

Establish the differential equation for each of the following primitives. (general solution) :

9. $y = Cx + C^2$. $[y = xy' + (y')^2]$

10. $\dfrac{x}{A} + \dfrac{y}{B} = 1$. $[y'' = 0]$.

11. $y = A + Bx + Cx^2$. $[y''' = 0]$.

12. $y = A \cos 3t + B \sin 3t$ $\left[\dfrac{d^2y}{dt^2} + 9y = 0\right]$

13. $y = Ax + Bx^{-1}$. $(x^2y'' + xy' - y = 0)$

14. $s = C_1 e^{2t} + C_2 e^{-t}$ $\left[\dfrac{d^2s}{dt^2} - \dfrac{ds}{dt} - 2s = 0\right]$

15. $y = Ae^{-pt} \sin(\omega t + B)$ $\left[\dfrac{d^2y}{dt^2} + 2p\dfrac{dy}{dt} + (p^2 + \omega^2)y = 0\right]$.

16. $y = \log(Ax)$ $\left[x\dfrac{dy}{dx} = 1\right]$

17. $r\theta = c_1 \cos \omega\theta + c_2 \sin \omega\theta$. $\left[\theta\dfrac{d^2r}{d\theta^2} + 2\dfrac{dr}{d\theta} + \omega^2 r\theta = 0\right]$.